$x > 0$, then ρ is uniquely determined.

Corollary 3.11. Let (S,d) be a metric space and $\{T_i\}$ $(i=1,2,\cdots)$ be a family of mappings of S into itself satisfying the condition: for each $i=1,2,\cdots$, $d(T_iq,T_ir) \le k\, d(q,r)$ for all q,r in S and for some k, $0<k<1$. If for all q in S, the sequence $\{T_iq\}$ converges to Tq and if $\rho_i \longrightarrow \rho$, where $\rho \in S$, then ρ is a fixed point of T and if T satisfies the condition such that there exists a constant k, $0<k<1$ such that $d(Tq,Tr) \le k\, d(q,r)$ for all q,r in S, then ρ is uniquely determined.

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THE WEAK ATTOUCH-WETS TOPOLOGY AND THE METRIC ATTOUCH-WETS TOPOLOGY

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ABSTRACT. The purpose of this paper is to find some relations between the weak Attouch-Wets topology and the metric Attouch-Wets topology for the nonempty closed convex subsets of a metrizable locally convex space X. We verify that the former is coaser than the latter. Moreover, we show that X is normable if and only if the two uniformities determining the two topologies for the closed convex subsets of $X \times R$ respectively are equivalent. Our results strengthen and sharpen those of Holá in terms of uniformity itself rather than the topology determined by the uniformity.

I. Introduction

As a successful generalization of the classical Kuratowski convergences of closed convex sets in finite dimensions [8], Attouch-Wets topology [1] in a general normed space X has lately attracted considerable attention. The reason why this topology receives a good deal of attention is that it is stable with respect to duality without reflexivity or even completeness. This Attouch-Wets topology is the topology of uniform convergence of distance functionals on bounded subsets of X, and is well suited for approximation and convex optimization. Its rich developements can be found in the literature[2][4][5].

Recently, Beer [3] defined, in the context of a locally convex space, the

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weak Attouch-Wets topology and the strong Attouch-Wets topology for the nonempty closed convex subsets. These topologies are, in general, different. In fact, it is essentially only in the normed setting that we get the same topology (see [3, Theorem 4.13]). One [3, Theorem 4.9] of his main theorems tells us that the strong convergence of a net of continuous linear functionals on Hausdorff locally convex space X can be explained in terms of the convergence of the corresponding net of its graphs in $X \times R$ with respect to the weak Attouch-Wets topology for the closed convex subsets $C(X \times R)$ or $X \times R$.

On the other hand, Holá [6] considered a "metric" Attouch-Wets topology for the closed convex subsets of a metrizable locally convex space, equipped with a translation invariant metric d. By an delementary method in functional analysis, he has shown that the metric Attouch-Wets convergence of graphs of linear functionals is stronger than convergence of the functionals in the strong topology, and that two notions coincide if and only if X is normable.

When X is a metrizable locally convex space with a translation invariant metric d, there are two topologies, namely, the weak Attouch-Wets topology and the metric Attouch-Wets topology for the nonempty closed convex subsets of X. In that case, it is natural to ask what the relation between the two topologies is. In the present paper, we will show that the latter is stronger than the former [Theorem 1]. Moreover, X is normable if and only if the two topologies for the nonempty closed convex subsets [Theorem 2]. Our results strengthen and sharpen those of Holá [Theorem 3 and 4] in terms of uniformity itself rather than the topology determined by the uniformity.

2. Preliminaries

We mainly refer to Beer [3]. As mentioned in the introduction, if X is a normed space, then the Attouch-Wets topology τ_{AW} on the nonempty closed convex subsets C(X) is the topology of uniform convergence of distance functionals on bounded subsets of X. As is well-known, the Attouch-Wets topology τ_{AW} can be presented as a uniform space. There are two standard uniformities representing τ_{AW}. A weaker uniformity determining τ_{AW} has a base consisting of all sets of the form

$$\{(A,C)\,|\,A\cap B\subset C+\varepsilon U \text{ and } C\cap B\subset A+\varepsilon U\}$$

where U is the solid unit ball of X, B is a bounded subset of X, and $\varepsilon > 0$. Motivated by this, Beer [3, Definition, p.7] gave the following definition in the locally convex setting.

Let X be a locally convex space. *The weak Attouch-Wets topology* τ_{AW}^{W} on $C(X)$ is the topology determined by the uniformity with typical basic entourages of the form

$$\Omega(B,U)=\{(A,C)\,|\,A\cap B\subset C+U \text{ and } C\cap B\subset A+U\}$$

where B is a closed bounded balanced convex subset and U is a convex balanced neighborhood of the origin.

Now we turn our attention to the metric space setting. Let (X,d) be a metrizable space with a compatible metric d. For $x_0\in X$ and $\varepsilon > 0$, $S_d[x_0,\varepsilon]$ denotes the open d-ball with center x_0 and radius $\varepsilon > 0$, and $S_d[A,\varepsilon]=\cup_{a\in A}S_d[a,\varepsilon]$ does the ε-parallel body for a subset A of X. Let $CL(X)$ be the nonempty closed subsets of X. *The Attouch-Wets topology* $\tau_{AW}(d)$ on $CL(X)$ is presented by a uniformity Σ_d which has a countable base consisting of all sets of the form

$$U_d[x_0,n]=\{(A,C)\,|\,A\cap S_d[x_0,n]\subset S_d[C,\tfrac{1}{n}] \text{ and } C\cap S_d[x_0,n]\subset S_d[A,\tfrac{1}{n}]\}$$

where x_0 is a fixed but arbitrary point of X and $n\in Z^+$. In particular, if X is a metrizable locally convex space with a translation invariant (in short, invariant) metric d, the relativized Attouch-Wets topology $\tau_{AW}(d)$ on $C(X)$ the nonempty closed convex subsets is called *the "metric" Attouch-Wets topology* in this paper.

In the sequel, X will be a metrizable locally convex space with an invariant metric d, X^* its continuous dual, and $\mathcal{U}$ will be the family of convex balanced neighborhoods fo the origin θ. The product $X\times R$ will be understood to be equipped with the box metric, denoted by $d \times |\cdot|$. Also we denote by $C(X)$ the nonempty closed convex subsets of X. Let us write $BC(X)$ for the family of all closed, bounded, balanced convex subsets of X.

3. Main Results

A set E in X is *bounded* if, for every neighborhood V of θ, we have $E \subset tV$ for all sufficiently large t. A set $E \subset X$ is said to be *d-bounded* if there is a number $M < \infty$ such that $d(x,y) \leq M$ for all x and y in E. In general, the bounded sets and the d-bounded ones need not be the same, even if d is invariant. If X is a normed space and d is the metric induced by the norm, then the two notions of boundedness coincide; but if d is replaced by $d_1 = d/(1+d)$, (an invariant metric which induces the same topology) they do not. However, we always assert the following.

Lemma. *Let X be a metrizable locally convex space with an invariant metric d. Then the family of d-bounded subsets contains the family of bounded ones.*

Proof. Let E be bounded but not d-bounded. We may choose a sequence $\{x_n\}$ in E satisfying $d(\theta, x_n) \geq n^2$. Since d is invariant, we have

$$d(\theta, nx) \leq nd(\theta, x)$$

for every $x \in X$ and for $n = 1, 2, 3, \cdots$. Taking $x = x_n/n$, we obtain

$$\frac{1}{n} d(\theta, x_n) \leq d(\theta, \frac{x_n}{n}).$$

Hence $d(\theta, x_n/n) (\geq n)$ does not tend to zero. Since d is a compatible metric, this implies x_n/n is not convergent to the origin θ. This contradicts the boundedness of E([9, Theorem 1.30, p.22]).

This simple lemma plays the crucial role in our results.

Theorem 1. *Let X be metrizable locally convex space with an invariant metric* *d. Then the uniformity* Σ_d *determining* τ_{AW}^{W} *for* $C(X)$ *is stronger than the one doing* τ_{AW}^{W} *for* $C(X)$. *Therefore* τ_{AW}^{W} *is coaser than* $\tau_{AW}(d)$.

Proof. It is sufficient to verify that every basic entourage $\Omega(B,U)$ contains some $U_d[\theta,n]$ in Σ_d where $B\in BC(X)$ and $U\in\mathcal{U}$. Since B is bounded, by Lemma there is an $n_0\in Z^+$ such that $B\subset S_d[\theta,n_0]$. The family $\{S_d[\theta,1/n]\}_{n=1}^{\infty}$ is a local base of the origin θ, so we may assume that $S_d(\theta,1/n_0]\subset U$. Observe that for a subset $E\subset X$ and $r>0$, we have $S_d[E,r]=E+S_d[\theta,r]$ because d is invariant. Then for $A,C\in C(X)$ we have

$$A\cap S_d[\theta,n_0]\subset S_d[C,\frac{1}{n_0}] = C+S_d[\theta,\frac{1}{n_0}]\Longrightarrow A\cap B\subset C+U$$

$$C\cap S_d[\theta,n_0]\subset S_d[A,\frac{1}{n_0}] = A+S_d[\theta,\frac{1}{n_0}]\Longrightarrow C\cap B\subset A+U.$$

Thus $U_d[\theta,n_0]\subset\Omega(B,U)$ as desired. Therefore, τ_{AW}^{W} is weaker than $\tau_{AW}(d)$.

As a direct consequence, we obtain the following;

Corollary. (Holá [6, Theorem 3]) Let $\{f_n\}$ be a net X^* and let $f\in X^*$. The τ_{AW} $(d\times|\cdot|)$-convergence of Grf_n to Grf implies that f_n is convergent to f in the strong topology. Here Grf denotes the graph of f in $X\times R$.

Proof. By Theorem 1, $Gr\,f_n$ converges to $Gr\,f$ in the weak Attouch-Wets topology τ_{AW}^{W} for $C(X\times R)$. Moreover, τ_{AW}^{W}-convergence is equivalent to the strong convergence of f_n to f in virtue of Beer's result [3, Theorem 4.9]. This forces us to get the result.

Remark. In the meantime, we provided a simple proof for 'Hola's result [6, Theorem 3].

Theorem 2. *X is normable if and only if the two uniformities* $\{\Omega(B,U)\}$ *and* Σ_{dx} *determining* τ_{AW}^{W} *and* $\tau_{AW}(d\times|\cdot|)$ *for* $C(X\times R)$ *respectively are equivalent (If X is a normed space, we take* $d=\|\cdot\|$ *the norm).*

Proof. If X is normed space and d is the metric induced by the norm $\|\cdot\|$, the box metric $d\times|\cdot|$ is norm (easily checked). Hence the boundedness and the $d\times|\cdot|$-boundedness on the normed space $(X\times R, d\times|\cdot|)$ coincide. Recall that the ball S_{dx} $[\theta,n]$ is convex balanced in this case. It is direct from these and Theorem 1 that the two uniformities $\{\Omega(B,U)\}$ and Σ_{dx} for $C(X\times R)$ are equivalent. Conversely, if the two uniformities are equivalent, then τ_{AW}^{W} and $\tau_{AW}(d\times|\cdot|)$ for $C(X\times R)$ are the same. Thus, the strong convergence of a net $\{f_n\}$ to f in X^* coincides with the $\tau_{AW}(d\times|\cdot|)$-convergence of its graphs by mearns of Beer's result [3, Theorem 4.9]. By Hola's result [6, Theorem 4], X is normable. This completes our proof.

Remark. Theorem 2, in fact, is a strengthened form of Hola's theorem [6, Theorem 4].

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