Journal of the Research Institute of Basic Sciences,
Korea Maritime University, Vol. 4, 1994.

Most Stringent Somewhere Most Powerful Tests under the Order Restrictions

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1. Notation and Definitions

We use the following notation :

$$\Phi(X) = \int_{\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2})du;$$

$$\Phi^X(X) = 1 - \Phi(X);$$

$$u_\alpha = \{\Phi^X\}^{-}1(\alpha);$$

$$\Phi^X(u_\alpha) = \alpha;$$

Further $t_{f:\alpha}$ is the solution of $P\{T_f \geq t_{f:\alpha}\} = \alpha$ where T_f has Student's t-distribution with f degrees of freedom. R^d will denote a linear subspace of dimension d (through the origin) in the sample space R^n. A superscript is used sometimes in order to denote that the corresponding symbol has one, and only one, fixed value.

We recapitulate some definitions, most of which are generally used. For that purpose, let (H, K) be a hypothesis testing problem, with composite alternative K.

Let $\beta_\Phi(\theta) = E_\theta\{\Phi(X)\}$ denote the power in θ of the test Φ.

Definition 1.1.1. The test Φ is of size α, if $\sup_{\theta\in H}\beta_\Phi(\theta) \le \alpha$ and is similar of size α, if $\beta_\Phi(\theta) = \alpha$ for all $\theta \in H$ and is unbiased of size α, if $\sup_{\theta\in H}\beta_\Phi(\theta) \le \alpha$ and $\inf_{\theta\in K}\beta_\Phi(\theta) \ge \alpha$, where $\theta \in H$ and $\theta \in K$ means that θ contains in the subset of the parametric space, restricted by the hypothesis H and K, respectively.

Definition 1.1.2. The envelope power function $\beta^*_D(\theta)$ of a class D of tests Φ, is determined by $\beta^*_D(\theta) = \sup_{\Phi\in D}\beta_\Phi(\theta)$.

Definition 1.1.3. The shortcoming $\gamma_{\Phi,D}(\theta)$ of a test Φ with respect to the class D, is defined by $\gamma_{\Phi,D}(\theta) = \beta^*_D(\theta) - \beta_\Phi(\theta)$.

Definition 1.1.4. A test Φ in a class C of tests, is said to be most stringent in C with respect to D for testing against K, if test Φ minimizes in C the maximum shortcoming with respect to D on the alternative K: $\sup_{\theta\in K}\gamma_{\Phi,D}(\theta) = \inf_{\psi\in C}\sup_{\theta\in K}\gamma_{\Phi,D}(\theta)$.

In the special case $C = D$, we obtain the most stringent (D) test. On specializing further, we obtain the most stringent size- α test in case D is the class of size-α tests.

The minimax principle leading to the proceeding definitions is sometimes quite unreasonable, a minimized maximum shortcoming on K often going with a large shortcoming for many alternatives $\theta \in K$. This objection to the two foregoing principles seems to be realistic for many problems of the form (H, K_1) and (H, K_2) that will presently be considered. (For these problems, it seems reasonable to restrict our attention to the subclass C of D, containing the tests Φ which have the shortcoming $\gamma_{\Phi,D}(\theta)$ equal to zero for some θ in K.

Definition 1.1.5. The tests Φ are said to be somewhere most powerful with respect to the class D, (abbreviated: S.M.P.(D)) if there is a θ in K such that $\gamma_{\Phi,D}(\theta) = \beta^*_D(\theta) - \beta_\Phi(\theta) = 0$.

By specializing the definition of "most stringent in C with respect to D for testing against K", in case C is the class of S.M.P.(D) tests, we obtain the most stringent S.M.P.(D) test. The following special cases will be applied:

(A) D is the class of size- α tests; we obtain the most stringent S.M.P. size- α test;

(B) D is the class of similar size- α tests; we obtain the most stringent S.M.P. similar size- α test;

(C) D is the class of unbiased size- α tests; we obtain the most stringent S.M.P. unbiased size- α test.

Obviously the most stringent S.M.P.(D) test has in general a larger maximum shortcoming than the most stringent (D) test, whereas the latter test has a larger shortcoming generally in a region inside the alternative. For many problems of the form (H, K_1) and (H, K_2) which will be considered, no clear cut preference will exist for either one of the two principles mentioned above.

2. The Formulation of the Problem

Let $X = (X_1, \cdots, X_n)$ have the multivariate normal distribution $N(\xi, \Sigma)$ with probability density function

$$f_X(X_1, \cdots, X_n) = \frac{|A|^{\frac{1}{2}}}{(2\pi\sigma^2)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{n} a^{ij}(X_i - \xi_i)(X_j - \xi_j)\right\}, \tag{2.1}$$

where the matrix A is nonsingular and known; $\Sigma = \sigma^2 A^{-1}$.

Problem will be considered where σ^2 is known, in which case we take σ^2 to be equal to 1, and also where σ^2 is unknown. The outcomes $X = (X_1, \cdots, X_n)$ of the random vector X and the vector $\xi = (\xi_1, \cdots, \xi_n)$ of means, can be regarded as points in the same n dimensional space R^n.

The vector ξ of means is known to lie in a subset of a given s-dimensional hyperplane V^s in R^n $(s \leq n)$ defined by the $(n-s)$ inequalities

$$b^{0h} + \sum_{i=1}^{n} b^{ih} = 0 \quad (h = 1, \cdots, n-s) \tag{2.2}$$

and the matrix form is

$$\begin{pmatrix} b^{01} & b^{11} & \cdots & b^{n1} \\ b^{02} & b^{12} & \cdots & b^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ b^{0n-s} & b^{1n-s} & \cdots & b^{nn-s} \end{pmatrix} \begin{pmatrix} 1 \\ \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where the coefficient matrix $[b^{ih}]$ $(i = 1, \cdots, n; h = 1, \cdots, n-s)$ is rank $n-s$.

The hypothesis H is to be tested that ξ lie in a given $(s-r)$ dimensional hyperplane V^{s-r} in V^s, defined by

$$\text{Hypothesis} \quad H : b^{0h} + \sum_{i=1}^{n} b^{ih}\xi_i = 0 (h = n-s+1, \cdots, n-s+r),$$

where $1 \leq r \leq s; [b^{ih}]$ $(i = 1, \cdots, n; h = 1, \cdots, n-s+r)$ is a matrix of rank $n-s+r$ and also the above hypothesis is presented by

$$\begin{pmatrix} b^{01} & b^{11} & \cdots & b^{n1} \\ \vdots & \vdots & \ddots & \vdots \\ b^{0n-s} & b^{1n-s} & \cdots & b^{nn-s} \\ \vdots & \vdots & \ddots & \vdots \\ b^{0n-s+r} & b^{1n-s+r} & \cdots & b^{nn-s+r} \end{pmatrix} \begin{pmatrix} 1 \\ \xi_1 \\ \cdot \\ \cdot \\ \xi_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

We shall first derive tests for H against the following one-sided alternative :

$$K_1 : b^{0h} + \sum_{i=1}^{n} b^{ih}\xi_i \geq 0 \quad (h = n-s+1, \cdots, n-s+r)$$

with at least one inequality strong; corresponding to a subset of V^s, which subset will be denoted also by K_1.

We shall describe some transformations simplifying the formulation of the problem defined above. We state here that the results of our investigation will be put in forms which do not depend on the particular transformations used,so the theory can be applied without an explicit construction of these transformations.

First, we choose the origin of R^n in V^{s-r} defined by the hypothesis H, thus obtaining a problem where

$$b^{0h} = 0 \quad (h = 1, \cdots, n-s-r) \tag{2.3}$$

holds true. So we can assume (2.3) in what follows. In this case all hyperplane V^t become linear subspaces R^t, containing the origin.

Next, the problem can be written in a simple form, by introducing a new basis in R^n. Denoting the points $(X_1, \cdots, X_n)$ of R^n by x, we define an inner product in R^n by means of the bilinear form

$$(x, y) = \sum_{i=1}^{n}\sum_{j=1}^{n} a^{ij} X_i Y_j = x'Ay = x' \sum^{-1} y. \tag{2.4}$$

Orthogonality $x \perp y$ is defined by $(x, y) = 0$, the norm $\|x\|$ is defined by

$$\|x\|^2 = (x, x)$$

and the metric is defined by

$$d(x,y) = |x-y| = (x-y)'A(x-y).$$

We can construct an orthonomal basis $f_1, \cdots, f_n$ for R^n, e.g., by using the Gram-Schmidt orthogonalization process, such that $f_{n-(s-r)+1}, \cdots, f_n$ span the linear subspace R^{s-r} defined by the hypothesis H and $f_{n-s-r}, \cdots, f_n$ span R^s defined by the Equalities (2.2). The problem can be reformulated by means of the coordinates $Y_1, \cdots, Y_n$ with respect to the basis $f_1, \cdots, f_n$, of the sample point $X : Y_i = (X, f_i) = \sum_{k=1}^{n}\sum_{l=1}^{n} f_{lk}a^{kl}x_l$ and by means of the new coordinates $\eta_1, \cdots, \eta_n$ of the vector ξ of mean value. Now we introduce the new quantities with respect to the above orthonomal basis $f_1, \cdots f_n$. Since $Y_i = (X, f_i)$, we have

$$E(Y_i) = E(X, f_i) = (EX, f_i) = (\xi, f_i) = \eta_i.$$

Since

$$Y_i = (X, f_i) = (f_i, X) = f_i'AX,$$

We have

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} f_1'AX \\ \vdots \\ f_n'AX \end{pmatrix} = \begin{pmatrix} f_1' \\ \vdots \\ f_n' \end{pmatrix} AX = \left(\begin{pmatrix} f_1' \\ \vdots \\ f_n' \end{pmatrix} A \right) X = (FA)X$$

considering the identities

$$FAF' = I_n, (f_i, f_i) = 1 = f_i'Af_i = \sum_\mu \sum_\nu a_{\mu\nu} f_i^\mu f_j^\nu$$

and

$$(f_i, f_j) = f_i'Af_j = 0.$$

Therefore y is equal to $(FA)x$ and η is EY, that is,

$$\eta = EY = E((FA)X) = FAEX = FA\xi,$$

and hence

$$\eta = (FA)\xi.$$

Next the covariance matrix of Y is

$$\begin{aligned} E(Y-\eta)(Y-\eta)' =& E(FAX - FA\xi)(FAX - FA\xi)' \\ =& FAE(X-\xi)(X-\xi)'(FA)' \\ =& FA\{\sigma^2 A^{-1} A' F'\} \\ =& \sigma^2 FAF' \\ =& \sigma^2 I_n \end{aligned}$$

The original coordinates $X_1, \cdots, X_n$ of the sample point X having the normal $N(\xi, \sum)$ distribution (2.1), the new coordinates $Y_1, \cdots, Y_n$ of X will have independent normal $N(\eta_i, \sigma^2)$ distribution given by probability density function

$$\frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \eta_i)^2\right\} \tag{2.5}$$

Since

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a^{ij}(X_i - \xi_i)(X_j - \xi_j) = \|X - \xi\|^2 = \sum_{i=1}^{n}(Y_i - \eta_i)^2.$$

Theorem 1.2.1. Let $X = (X_1, \cdots, X_n)$ be distribution to $N(\xi, \sum)$, where $\sum = \sigma^2 A^{-1}$. Transform $Y_i = (X, f_i)$ according to the above new orthonormal basis, then $Y = (Y_1, \cdots, Y_n)$ is distributed to the independent normal distribution $N(\eta, \sigma^2 I)$, where $\eta = FA\xi$.

The vector ξ of means $\eta_1, \cdots, \eta_n$ is known to lie in the s-dimensional linear subspace R^s, defined by the equalities

$$\eta_i = 0 \quad (i = 1, \cdots, n-s) \tag{2.6}$$

and the hypothesis H is to be tested, that ξ lies in the subspace R^{s-r}, defined by

$$\text{Hypothesis } H : \eta_i = 0 \quad (i = n-s+1, \cdots, n-s+r),$$

whereas the alternative K_1 becomes a subset in R^s of the form

$$K_1 : \sum_{i=n-s+1}^{n-s+r} d^{ih}\eta_i \geq 0 \quad (h = 1, \cdots, r)$$

with at least one inequality strong; where

$$[d^{ih}] \ (i = n-s+1, \cdots, n-s+r; h = 1, \cdots, r)$$

is a matrix of rank r.

3. Problem $\{(H, K_1), \sigma^2$ is unknown $\}$

By arguments similar to those of Problem $\{(H, K_1), \sigma^2 = 1\}$, we can derive the most stringent somewhere most powerful size-α test for Problem $\{(H, K_1), \sigma^2$ is unknown $\}$.

By applying an obvious modification of Theorem 1 in [7] (p. 161), we obtain the uniformly most powerful (abbreviated:(U.M.P)) similar size-α test Φ:

$$\frac{OX^I}{\|X^I - X^{II}\|} \geq \frac{t_{n-(s-r)-1;\,\alpha}}{(n-(s-r)-1)^{\frac{1}{2}}} \tag{3.1}$$

for Problem $\{(H, A_1)$, σ^2 is unknown $\}$ where the alternative

$$A_1 : \eta_i = 0 \quad (i = 1, \cdots, n-s)$$

$$\eta_i = \theta\eta_i^{(1)} \quad (i = n-s+1, \cdots, n-s+r), \theta > 0,$$

Let X^I is the projection of the sample point X onto the R_1 spanned by the half-line $l = A_1'$, X^{II} is the projection of the sample point X onto the R^{n-s+r} perpendicular to the R^{s-r} defined by the hypothesis H, in the sample space R^n. So the class C of somewhere most powerful similar size-α tests for Problem $\{(H, K_1\}$, σ^2 is unknown $\}$ is determined by (3.1) for l varying over K_1'.

We observe that the critical region belong to the test (3.1) consists of the points whose orthogonal projections onto R^{n-s+r} are inner or boundary points of a semi-cone of revolution with axis l and semi-angle

$$\Lambda_1 = \cot^{-1}\left\{(n-s+r-1)^{-\frac{1}{2}} t_{n-s+r-1}; \alpha\right\}. \tag{3.2}$$

It can be proved that the maximum shortcoming $\sup_{\theta\in m}\gamma_{\theta,C}(\theta)$ of Test (3.1) over the half-line m in K_1', is a non-decreasing function of $\Psi(l,m)$ which strictly increase for $\Psi(l,m) \leq \Lambda_1$ and is constantly equal to 1 for $\Psi(l,m) > \Lambda_1$. Since, we consider that $\beta_\Phi(\theta)$ of S.M.P similar size- α test for $\{(H, K_1), \sigma^2$ is unknown $\}$ is defined on the class C at power function Φ. Then $\beta_C^*(\theta)$ is invariant under translation parallel to R^{s-r} confined in K_l'. That is,

$$\sup_{\theta\in K_1}\gamma_{\Phi,D}(\theta) = \sup_{\theta\in K_1}\gamma_{\Phi,C}(\theta) = \sup_{m\subset K_1'}\{\sup_{\theta\in m}\gamma_{\Phi,C}(\theta)\}.$$

Hence

$$\beta_C^*(\theta) = \sup_{\Phi\in C}\beta_\Phi(\theta) = \beta_{\Phi_m}(\theta)$$

where Φ_m is U.M.P. tests on the half-line m in K_1'. Therefore

$$\sup_{\theta\in m}\gamma_{\Phi,C}(\theta)=\sup_{OQ>0}\{\beta_C^*(Q)-\beta_\Phi(Q)\}$$

$$=\sup_{OQ>0}\{\beta_{\Phi_m}(Q)-\beta_\Phi(Q)\}.$$

Actually the maximum shortcoming $\sup_{\theta\in m}\gamma_{\Phi,C}(\theta)$ over m in K_1', is nondecreasing function of $\Psi(l,m)$ which strictly increase for $\Psi(l,m)\leq\Lambda_1$ (Since $\beta_{\Phi_m}(Q)$ is fixed and $\beta_\Phi(Q)$ is decrease) and

$$\sup_{\theta\in m}\gamma_{\Phi,C}(Q)=\sup_{OQ>0}\{\beta_C^*(Q)-\beta_\Phi(Q)\}=1$$

if $\Psi(l,m)>\Lambda_1$.

Applying consideration to those of problem $\{(H,K_1),\sigma^2=1\}$, we obtain the following result.

Theorem 1.3.1. (A) In case $\Psi_0\leq\Lambda_1$, the most stringent somewhere most powerful similar size-α test Φ_0 for Problem $\{(H,K_1),\ \sigma^2$ is unknown $\}$ is determined by (3.1), taking for l the half-line l_0 satisfying

$$\sup_{m\subset K}\Psi(l_0,m)=\inf_{l\subset K_1'}\sup_{m\subset K_1'}\Psi(l,m)$$

(B) In case $\Psi_0>\Lambda_1$, each somewhere most powerful similar size-α test has the maximum shortcoming on K_1 equal to 1, so that no uniquely determined most stringent somewhere most powerful similar size -α test exists for Problem $\{(H,K_1),\ \sigma^2$ is unknown $\}$ in this case.

4. An Application

The theory Section 3 will be applied to the Problem of testing homogeneity of means ($\mu_1 = \mu_2 = \cdots = \mu_k$) against an upward trend ($\mu_1 \le \mu_2 \le \cdots \le \mu_k$, with at least one inequality strong), as mentioned at the end of introduction. This problem is of the form $\{(H, K_1)$, σ^2 is unknown $\}$, where X_ν corresponds with X_{ij} ($j = 1, \cdots, n_i$, $i = 1, \cdots, k$) distributed as $N(\mu_i, \sigma^2)$, σ^2 is unknown, when $\nu = \sum_{h=1}^{i} n_h + j$ ($\nu = 1, \cdots, n$). The indices $\nu = 1, \cdots, n$ are subdivided into k blocks of $n_1, n_2, \cdots, n_k$ indices respectively. We have

$$s = k; r = k-1; (x, y) = \sum_{\nu=1}^{n} X_\nu Y_\nu = \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij} Y_{ij}$$

and

$$R^s = R^k = \{(\mu_1 \cdots \mu_1 : \mu_2 \cdots \mu_i \cdots \mu_k)\},$$

which notation indicates that the points of R^s have coordinates which are equal within each block.

Similarly, we have always mentioning the j-th coordinate of i-th block;

$$R^{s-r} = R^{k-(k-1)} = R^1 = \{(\mu \cdots \mu : \mu \cdots \mu \cdots \mu)\}$$

$$R^r = R^{k-1} = \{(\mu_1 \cdots \mu_1 : \mu_2 \cdots \mu_i \cdots \mu_k)\},$$

where $\sum_{i=1}^{k} n_i \mu_i = 0$ has to be satisfied;

$$R^{n-s+r} = R^{n-(s-r)} = R^{n-1}$$

$$= \{X_{11}, \cdots, X_{1n_1}; X_{21}, \cdots, X_{ij}, \cdots, X_{kn_k})\},$$

where $\sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij} = 0$ has to be satisfied;

The $r = (k-1)$ edges e_g $(g = 1, \cdots, k-1)$ of k_1' in $R^r = R^{k-1}$ are determined by

$$e_g = \{(\mu_1 \cdots \mu_1; \mu_2 \cdots \mu_i \cdots \mu_k)\},$$

where

$$\mu_i = -\theta s_g^{-1}\ (i \leq g); \mu_i = \theta(n - s_g)^{-1}\ (i > g); \theta > 0,$$

with the notation:

$$s_g = \sum_{i=1}^{g} n_i\ (g = 1, \cdots, k),\ s_0 = 0. \tag{4.1}$$

The arbitrary half-line

$$l = \{(\theta\omega_1 \cdots \theta\omega_1; \theta\omega_2 \cdots \theta\omega_i \cdots \theta\omega_k)\}, \theta > 0 \tag{4.2}$$

is in K_1' provided that

$$\sum_{i=1}^{k} n_i\omega_i = 0\ (\omega_1 \leq \omega_2 \leq \cdots \leq \omega_k). \tag{4.3}$$

The angles $\Psi(l, e_g)$ $(g = 1, \cdots, k-1)$ are determined by

$$\Psi(l, e_g) = \cos^{-1}\{\frac{(l, e_g)}{||l||\,||e_g||}\}$$

$$= \cos^{-1}\left\{-\frac{n^{\frac{1}{2}} \sum_{i=1}^{g} n_i\omega_i}{(\sum_{i=1}^{k} n_i\omega_i^2)^{\frac{1}{2}} s_g^{\frac{1}{2}} (n - s_g)^{\frac{1}{2}}}\right\}$$

Since

$$||l||^2 = \theta^2({\omega_1}^2 + {\omega_2}^2 + \cdots + {\omega_k}^2),$$

then

$$\|l\| = \theta(\sum_{i=1}^{k} n_i\omega_i^2)^{\frac{1}{2}},$$

and

$$\begin{aligned}\|e_g\|^2 &= \left[\theta^2 \sum_{i=1}^{g} \frac{n_i}{s_g^2}\right] + \left[\theta^2 \frac{n-s_g}{(n-s_g)^2}\right] \\ &= \theta^2 \left\{\frac{s_g}{s_g^2} + \frac{n-s_g}{(n-s_g)^2}\right\} \\ &= \frac{\theta^2 n}{s_g(n-s_g)}\end{aligned}$$

then

$$\|e_g\| = \theta n^{\frac{1}{2}}\{s_g(n-s_g)\}^{-\frac{1}{2}}$$

Therefore

$$\begin{aligned}(l, e_g) &= -\theta^2 \sum_{i=1}^{g} n_i \frac{1}{s_g}\omega_i + \theta^2 \sum_{i=1}^{k} n_i \frac{1}{n-s_g}\omega_i \\ &= -\theta^2 \left[\sum_{i=1}^{g} \frac{n-s_g}{s_g(n-s_g)} n_i\omega_i - \sum_{i=g+1}^{k} \frac{s_g}{n-s_g} n_i\omega_i\right] \\ &= -\theta^2 \sum_{i=1}^{g} \frac{n}{s_g(n-s_g)} n_i\omega_i \\ &= -\theta^2 n \frac{1}{s_g(n-s_g)} \sum_{i=1}^{g} n_i\omega_i.\end{aligned}$$

Consequently, l is the half-line l_0, making equal angles

$$\Psi_0 = \cos^{-1}\{(\sum_{i=1}^{k} n_i\omega_i^2)^{-\frac{1}{2}}\} \tag{4.4}$$

with the edges e_g $(g = 1, \cdots, k-1)$, if the weights ω_i $(i = 1, \cdots, k)$ satisfy the equations

$$-n^{\frac{1}{2}} \sum_{i=1}^{g} n_i \omega_i = s_g{}^{\frac{1}{2}} (n - s_g)^{\frac{1}{2}} \ (g = 1, \cdots, k).$$

The solution

$$\omega_i = n^{-\frac{1}{2}} n_i^{-1} \{ S_{i-1}^{\frac{1}{2}} (n - s_{i-1})^{\frac{1}{2}} - s_i{}^{\frac{1}{2}} (n - s_i)^{\frac{1}{2}} \} \ (i = 1, \cdots, k) \tag{4.5}$$

of these equations satisfies the Inequalities (4.3). So $l_0 \subset K_1'$ and l_0 satisfies (3.3). The projection X^I of the sample point X on l_0 is determined by (4.2) where θ has to minimize

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \theta \omega_i)^2.$$

So

$$\frac{\partial}{\partial \theta} \left[\sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \theta \omega_i^2) \right]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{n_i} 2 (X_{ij} - \theta \omega_i)(-\omega_i) = 0.$$

Therefore

$$\begin{aligned} \theta &= (\sum_{i=1}^{k} \omega_i^2)^{-1} \sum_{i=1}^{n_i} \omega_i X_{ij} \\ &= (\sum_{i=1}^{k} n_i \omega_i^2)^{-1} \sum_{i=1}^{k} n_i \omega_i X_i; \end{aligned}$$

$$OX^I = \theta (\sum_{i=1}^{k} n_i \omega_i^2)^{\frac{1}{2}}, \tag{4.6}$$

(4.6) where X_i denotes the sample mean $n_i^{-1}\sum_{j=1}^{n} X_{ij}$.

The projection X^{II} of X on $R^{n-s+r} = R^{n-1}$ is defined by

$$X^{II} = (X_{11} - X_{..}, \cdots, X_{ij} - X_{..}, \cdots, X_{kn_k} - X_{..}),$$

where $X_{..} = n^{-1}\sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{ij}$, so we have

$$\begin{aligned}\|X^I - X^{II}\|^2 &= \|X^{II}\|^2 - \|X^I\|^2 \\ &= \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - X_{..})^2 - (OX^I)^2. \qquad (4.7)\end{aligned}$$

The condition $\Psi_0 \leq \Lambda_1$ (see (3.2) and (4.4)) can be written in the following form

$$t_{n-2;\alpha} \leq (n-2)^{\frac{1}{2}}(\sum_{i=1}^{k} n_i\omega_i^2 - 1)^{-\frac{1}{2}} \qquad (4.8)$$

Applying the Theorem 1.3.1, we obtain the following.

Corollary 1.4.1. The test

$$\frac{(\sum_{i=1}^{k} n_i\omega_i^2)^{-\frac{1}{2}}\sum_{i=1}^{k} n_i\omega_i X_{i.}}{\left\{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - X_{..})^2 - (\sum_{i=1}^{k} n_i\omega_i^2)^{-1}(\sum_{i=1}^{k} n_i\omega_i X_{i.})^2\right\}^{\frac{1}{2}}} \geq \frac{t_{n-2}}{(n-2)^{\frac{1}{2}}},$$

where the weights ω_i ($i = 1, \cdots, k$) are determined by (4.5) and (4.1) is the most stringent somewhere most powerful similar size-α k-sample test against an upward trend if (4.8) holds true.

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