Uniformly Most Powerful Unbiased Tests of the Exponential Parameter Based on Accelerated Life Testing

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Abstract

Developed in this paper is a uniformly most powerful unbiased test procedure for the mean of an exponential distribution based on the observations from accelerated life tests. Type I censoring with replacement is assumed at each overstress level. The formulated hypotheses are shown to be related with a parameter in an exponential family, and a conditional argument from Lehmann is utilized to develop an exact test procedure.

1. Introduction

Accelerated life tests are frequently employed in industry to reduce the amount of time required for measuring reliabilities of devices or components. Developed in this paper is procedure for testing $H_0 : \theta_u \geq \theta_0$, where θ_u is the mean lifetime of an exponential distribution at the use condition, against $H_1 : \theta_u < \theta_0$ based on accelerated life tests(ALT).

For the above hypothesis test, Lawless(1982) presents the likelihood ratio and normal approximation methods for censored samples based on asymptotic theories. For complete (uncensored) samples, Lawless also presents exact conditional procedures. In this paper, we assume Type I censoring(with replacement) at each overstress level, and develop a uniformly most powerful(UMP) unbiased test procedure based on the exact conditional arguments of Lehmann(1986).

2. The alt model

We assume that the lifetme T of a test unit at stress level s has an exponential distribution described by the following probability density function.

$$f(t\,;\,s)=(1/\theta_s)\,exp\,(-t/\theta_s),\quad t>0.$$

The mean lifetime θ_s and stress s are assumed to be related as

$$\theta_s=exp(\beta_0+\beta_1 s), \tag{1}$$

where β_0 and β_1 are unknown constants. relationship (1) is frequently used in ALT. In fact, it can be shown that the well – known inverse power law model or the Arrhenius reaction rate model is a special case of (1).

The proposed ALT involves $m(\geq 2)$ overstress levels. The high stress level s_m is usually specified on the basis of engineering judgement. Typically, there exists a limit on s_m, within which relationship (1) is regarded as appropriate. The other $(m-1)$ stress levels are chosen such that $s_u < s_1 < s_2 < \cdots < s_{m-1} < s_m$ where s_u is the use condition stress. Then, without loss of generality, we can standaridize the use and high stress levels to be 0 and 1, respectively, by the following transformation.

$$s = (s' - s'_u)/(s'_m - s'_u),$$

where a prime is used to denote the original scale of the stress. The other stress levels are standardized accordingly.

Concerning the life tests at the overstress levels, we assume the following.

1. $n_i (i = 1, 2, \cdots, m)$ test units are put on test at time 0 under the constant application of stress s_i.
2. The lifetimes of test units are independent.
3. Failed units are immediately replaced with new ones.
4. Life test at overstress level $i (i = 1, 2, \cdots, m)$ is terminated at time t_{ci}.

3. Uniformly most powerful unbiased test

Our objective is to test $H_0 : \theta_u \geq \theta_0$ against $H_1 : \theta_u < \theta_0$ based upon the data from the aforementioned accelerated life tests. Since $\theta_u = exp(\beta_0)$, the above hypotheses can be rewritten as

$$H_0 : \beta_0 \geq ln\theta_0,$$
$$H_1 : \beta_0 < ln\theta_0. \qquad (2)$$

Let r_i be the number of failures at overstress level s_i up to the censoring time $t_{ci} (i = 1, 2, \cdots, m)$. Then, by assumption, r_i's are independent Poisson random variables with paramenters $n_i t_{ci}/\theta_i$ respectively, where

$$\theta_i = exp(\beta_0 + \theta_1 s_i), \; i = 1, 2, \cdots m. \qquad (3)$$

Then, the joint distribution of $r_1, r_2, \cdots, r_m$ can be expressed as

$$f(r_1, r_2, \cdots, r_m)$$
$$= \prod_{i=1}^{m} \frac{1}{r_i!}\left(\frac{n_i t_{ci}}{\theta_i}\right)^{ri} \exp\left(-\frac{n_i t_{ci}}{\theta_i}\right)$$
$$= \left[\prod_{i=1}^{m} \exp\left(-\frac{n_i t_{ci}}{\theta_i}\right)\right] \exp\{-\beta_0 \sum_{i=1}^{m} r_i - \beta_1 \sum_{i=1}^{m} s_i r_i\}$$
$$\times \left[\prod_{i=1}^{m} \frac{(n_i t_{ci}}{r_i!}\right] \qquad (4)$$

The above joint distribution function belongs to the following multiparamenter exponential family.

$$f(\mathrm{x}) = D(\tau, \delta) exp\{\tau U(\mathrm{x}) + \sum_i \delta_i V_i(\mathrm{x})\} P(\mathrm{x}). \qquad (5)$$

Comparing(4) with(5), we have $\mathrm{x} = (r_1, r_2, \cdots, r_m)$, $r = -\beta_0$, $\delta = -\beta_1$, $U = \Sigma_{i=1}^{m} r_i$, and $V = \Sigma_{i=1}^{m} s_i r_i$, among others.

The hypotheses in (2) concern with parameter β_0 in an exponential family, with β_1 occurring as a nuisance parameter. Then, as Lehmann(1986) shows, conditional tests for β_0 with β_1 unspecified can be constructed, and these ar UMP unbiased tests. More specifically, the conditional distrbution of $U(=\Sigma_{i=1}^{m} r_i)$ given $V = \Sigma_i^m s_i r_i = v$ constitutes an one – parameter exponential family, and there exists a UMP unbiased level – α test for testing (2) with the following critical function.

$$\phi(u, v) = \begin{cases} 1 & if \; u > C(v) \\ \gamma(v) & if \; u = C(v) \\ 0 & if \; u < C(v) \end{cases}$$

where C and γ are determined by

$$P_r\{U > C(v) \mid v, \beta_0 = ln\beta_0\}$$
$$+ \gamma(v) P_r\{U = C(v) \mid v, \beta_0 = ln\beta_0\} = \alpha.$$

To determine C and γ for given v, we need to find

$$P_r\{U=u \mid V=v\}$$
$$=P_r\{U=u, V=v\}/P_r\{V=v\}. \qquad (6)$$

We first evaluate the numerator of (6) as follows.

$$P_r\{U=u, V=v\}$$
$$=P_r\{\sum_{i=1}^{m} r_i=u, \sum_{i=1}^{m} s_i r_i=v\}.$$

Define $R_{uv}=\{r=(r_1, r_2, \cdots, r_m)\}$ where r_i's are nonnegative integers such that $\Sigma_{i=1}^{m} r_i=u$ and $\Sigma_{i=1}^{m} s_i r_i=v$. Then, from(4)

$$P_r\{U=u, V=v\}$$
$$=\sum_{r\in R_{uv}} f(r_1, r_2, \cdots, r_m)$$
$$=\sum_{r\in R_{uv}}\left[\prod_{i=1}^{m}\exp\left(\frac{-n_i t_{ci}}{\theta_i}\right)\right]\exp(-\beta_0 u-\beta_1 v)$$
$$\times\left[\prod_{i=1}^{m}\frac{(n_i t_{ci})^{r_i}}{r_i!}\right]$$
$$=\exp(-\beta_0 u-\beta_1 v)\prod_{i=1}^{m}\exp\left(-\frac{n_i t_{ci}}{\theta_i}\right)$$
$$\times\sum_{r\in R_{uv}}\left[\prod_{i=1}^{m}\frac{(n_i t_{ci})^{r_i}}{r_i!}\right]. \qquad (7)$$

Next, consider the denominator of (6).

$$P_r\{V=v\}$$
$$=P_r\{\sum_{i=1}^{m} s_i r_i=v\}.$$

Define $R_v=\{r=(r_1, r_2, \cdots, r_m)\}$ where r_i's are nonnegative integers such that $\Sigma_{i=1}^{m} s_i r_i=v$. Note that $\Sigma_{i=1}^{m} r_i$ is not necessarily equal to u for $r\in R_v$. then.

$$Pr\{V=v\}$$
$$=\sum_{r\in R_v} f(r_1, r_2, \cdots, r_m)$$
$$=\exp(-\beta_1 v)\prod_{i=1}^{m}\exp\left(-\frac{n_i t_{ci}}{\theta_i}\right)$$
$$\times\sum_{r\in R_v}\left[\exp(-\beta_0\sum_{i=1}^{m} r_i)\prod_{i=1}^{m}\frac{(n_i t_{ci})^{r_i}}{r_i!}\right]. \qquad (8)$$

Combining (7) and (8), we have

$$P_r\{U=u \mid V=v\}$$
$$=\left[\exp(-\beta_0 u)\sum_{r\in R_{uv}}\prod_{i=1}^{m}\frac{(n_i t_{ci})^{r_i}}{r_i!}\right]$$
$$\times\left[\sum_{r\in R_v}\exp\left(-\beta_0\sum_{i=1}^{m} r_i\right)\prod_{i=1}^{m}\frac{(n_i t_{ci})^{r_i}}{r_i!}\right]^{-1}$$
$$=\left[\sum_{r\in R_{uv}}\prod_{i=1}^{m}\frac{(n_i t_{ci}/\theta_u)^{r_i}}{r_i!}\right]\left[\sum_{r\in R_v}\prod_{i=1}^{m}\frac{(n_i t_{ci}/\theta_i)^{r_i}}{r_i!}\right]^{-1},$$

where $\theta_u=exp(\beta_0)$. then, given that $V=v$ and $\theta_u=\theta_0$(or equivalently, $\beta_0=ln\theta_0$), we first determine $C(v)$ as a positive integer such that

$$P_r\{U>C(v) \mid v, \theta_0\}<\alpha,$$
$$P_r\{U\geq C(v) \mid v, \theta_0\}>\alpha. \qquad (9)$$

Next, $\gamma(v)$ is determined as

$$\gamma(v)=\frac{\alpha-P_r\{U>C(v) \mid v, \theta_0\}}{P_r\{U=C(v) \mid v, \theta_0\}} \qquad (10)$$

The power of the above conditional test at $\theta_u=\theta_1<\theta_0$(or equivalently, at $\beta_0=ln\theta_1$) is given by

$$Power(\theta_1 \mid v)$$
$$=P_r\{U>C(v) \mid v, \theta_1\}+\gamma(v)P_r\{U=C(v) \mid v, \theta_1\}. \qquad (11)$$

Lehmann(1986) states that $Power(\theta_1 \mid v)$ may be considered as an estimate of the unconditional power.

As an illustration, suppose that we want to test hypotheses(2) with $\theta_0=1,000$(hrs) and $\alpha=0.05$. Assume that accelerated life tests were conducted at $s_1=0.3$, $s_2=0.6$, and $s_3=1$ with $n_1=20$, $n_2=15$, $n_3=10$ and $t_{c1}=t_{c2}=t_{c3}=50$(hrs). Suppose $r_1=2$, $r_2=4$, and $r_3=7$ were observed. Then, from (9) and (10) we obtain $C(v)=15$ and $\gamma(v)=0.6388$, respectively. since $u=r_1+r_2+r_3=13<C(v)$, $H_0 : \theta_u\geq\theta_0$ cannot be rejected. The power of the above conditional test against the alternative $H_1 : \theta_u=\theta_1$ can be calculated using (11). For instance, when $\theta_1=200$(hrs) the conditional power becomes 0.8228.

4. Concluding remarks

Although the developed test procedure possesses certain optimal properties, it cannot be used to plan accelerated life teste(i.e., to determine n_i, t_{ci}, etc.) since the conditional power at $\theta_u = \theta_1 < \theta_0$ can be calculated only after the observations become available. However, as stated by Lehmann(1986), a procedure guranteeing a specified power P^* against $H_1 : \theta_u = \theta_1 < \theta_0$ can be obtained by continuing taking observations until $Power(\theta_1 \mid v) \geq P^*$.

We also investigated the possibility of applying the present method for constructing(conditional) UMP unbiased tests to such accelerated life test situations as Type I censoring without replacement and Type II censoring with or without replacement. However, the corresponding distributions of random variables cannot be reduced to the exponential family in (5), and therefore, the present method may not apply to those testing situations.

For the present ALT, the maximum likelihood estimation procedure for β_0 and β_1 is described in Appendix. For the case of two overstress levels, the maximum likelihood estimates of β_0 and β_1 can be determined in closed forms. developing an exact unconditional test for β_0 based upon $\hat{\beta}_0$ and/or $\hat{\beta}_1$ and comparing it with the present conditional method may be a fruitful area of future rescarch. It is worth noting that Yum and Kim(1990) have recently developed an exact unconditional test procedure for β_0 when accelerated life tests are conducted under Type II censoring with or without replacement.

5. Appendix

For the ALT described in the second section, the likelihood function is given by(see Bain (1978, p.155) for the likelihood function in the case of single stress level)

$$L' = \prod_{i=1}^{m} \left(\frac{n_i}{\theta_i} \right)^{r_i} \exp(-n_i t_{ci} / \theta_i).$$

Taking logarithm of L' and using (3), we have

$$\begin{aligned} L = \ln L' &= \sum_{i=1}^{m} \{ r_i (in\ n_i - \ln \theta_i) - n_i t_{ci} / \theta_i \} \\ &= \sum_{i=1}^{m} \{ r_i (in\ n_i - \beta_0 - \beta_1 s_i) \\ &\qquad - n_i t_{ci} \exp(-\beta_0 - \beta_1 s_i) \}. \end{aligned}$$

Then, the likelihood equations are given by

$$\frac{\partial L}{\partial \beta_0} = \sum_{i=1}^{m} \{ -r_i + n_i t_{ci} \exp(-\beta_0 - \beta_1 s_i) \} = 0$$
$$\frac{\partial L}{\partial \beta_1} = \sum_{i=1}^{m} \{ -r_i s_i + s_i n_i t_{ci} \exp(-\beta_0 - \beta_1 s_i) \} = 0,$$

which can be reduced to

$$\exp(-\beta_0) \sum_{i=1}^{m} n_i t_{ci} \exp(-\beta_1 s_i) = \sum_{i=1}^{m} r_i$$
$$\exp(-\beta_0) \sum_{i=1}^{m} s_i n_i t_{ci} \exp(-\beta_1 s_i) = \sum_{i=1}^{m} s_i n_{i.} \qquad (A1)$$

For the case where two overstress levels are involved, equations(A1) yield closed form estimates of β_0 and β_1 as follows.

$$\hat{\beta}_0 = \frac{1}{1 - s_i} \left[s_1 \ln\left(\frac{r_2}{n_1 t_{c2}} \right) - \ln\left(\frac{r_1}{n_i t_{c1}} \right) \right]$$
$$\hat{\beta}_1 = \frac{1}{1 - s_i} \left[\ln\left(\frac{r_1}{n_1 t_{c1}} \right) - \ln\left(\frac{r_2}{n_2 t_{c2}} \right) \right].$$

If either $r_1 = 0$ or $r_2 = 0$, then acceptable solutions to (A1) do not exist. When there exist more than two overstress levels, an iterative method(e.g., Newton – Raphson method) may be used to determine $\hat{\beta}_0$ and $\hat{\beta}_1$.

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