Journal of the Research Institute of Basic Sciences,
Korea Maritime University, Vol. 2, 1992.

Regularity for solutions of Transmission Problems for Elliptic Equations with Internal Polygonal Boundary

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1. INTRODUCTION

Let D be a bounded in $R^n(n\geq 2)$and let $D\subset B_{1/2}$ where B_r is the ball at the center 0 with the radius r. Denote $B_1 = B$. Consider a weak-solution $u\in W^{1,2}(B)$, the space of square integrable function with distributional derivatives in $L^2(B)$, of the discontinuous elliptic equation:

(0.1)
$$\sum_{i,j=1}^{n}\frac{\delta}{\delta x_i}\left(a_{ij}(x)\frac{\delta u}{\delta x_i}\right)=0 \quad \text{in } B$$

where

$$\begin{cases} a_{ij}(x) = a^1_{ij}\,XD + a^2_{ij}XR^n\backslash D \\ (a^1_{ij}) \text{ and } (a^2_{ij}) \text{ are positive definite symmetric constant matrices.}\end{cases}$$

We are interested in establishing a regularity estimates

(1.2)
$$\int_{\delta D}|(\nabla u\pm)^*|^2 d\sigma < \infty$$

where

$$(g^+)^*(Q) = \sup\{|g(X)|:|X-Q|\leq(1+\alpha)\text{dist}(X,\partial D),\ X\in D\}$$

and

$$(g^-)^*(Q) = \sup\{|g(X)|:|X-Q|\leq(1+\alpha)\text{dist}(X,\partial D),\ X\in B_{3/4}\backslash D\}$$

In [L,R,U] and [R,S,U], where ∂D is locally the graph of a function in he class $W^{p,n}$ and $p > n$ the gradient of u is Holder continous up to the boundary on the both sides of D.
But their method does not work for non smooth domain D.
Escauriaza, Fabes and Verchota [E,F,V] established (1.2) for any bounded Lipschitz domains with a connected boundary when $(a^1{}_{ij})=I$ and $(a_{ij})^2 = kI$. Here I denote the indenity matrix.
Escauriaza and Seo [E,S] have established (1.2) for any bouneded Lipschitz domain provided$(a^1{}_{ij}-a^2{}_{ij})$ is positive or negative semidefinite matrix . They also established a similar results for solutions to transmission problems for a elasticity and paraboic equations. But the method they used to obtain(1.2) does not work in all the situations.
We are now interested in removing the semi-postiveness restrictions to the difference of coefficient matrices$(a^1{}_{ij}-a^2{}_{ij})$[E,S]. In this paper, we get a partial result in R^2 under some severe geometric restriction.
Throughout this section we assume that the internal domain D satisfies

(1.3) $$D \cap B_{3r_0} = \{(r,\theta):0 \le r < 3r_0,\ -\theta_0 < \theta < \theta_0\}$$

so that $\partial D \cap B_{3r_0}$ consists of line segments

$$l_1 = \{(r,\theta_0):0 \le r < r_0\} \quad \text{and}$$
$$l_2 = \{(r,-\theta_0):0 \le r < r_0\}.$$

THEOREM 1. *If* u is a week-solution of the transmission problem:

(1.4) $$\frac{\partial}{\partial x_i}\Big[((a-1)\chi D+1)\frac{\partial}{\partial x_i}(x,\ y) \Big] + \frac{\partial}{\partial x_i}\Big[((a-1)\chi D+1)\frac{\partial}{\partial x_i}(x,\ y) \Big]=0$$

in B_{2r_0}, for any $a > 0$ and $d > 0$ then we have the estimate

(1.5) $$\int_{(l1 \cup l2)} |(\nabla u\pm)^*|^2 d\sigma < \infty$$

Since $D \cap B_{2r_0}$ is symmetric with respect to x-axis, we may use the following useful properties:
SYMMETRY PROPERTIES. *If* u is a weak-solution of (1.4) in B_{3r_0}, then
1. $\tilde{u}(x,y) \overset{\text{def}}{=} u(x,-y)$ is also a solution of (1,4)
2. $v(x,y) \overset{\text{def}}{=} u(x,y) + \tilde{u}(x,y)$ and

$v(x,y) \overset{\text{def}}{=} u(x,y) - \tilde{u}(x,y)$ are again solution of (1.4)

3. $v(x,0) = 0$ and $\frac{\partial w}{\partial y}(x,0) = 0$
4. for$(x,y) \in l_1$.

$$\left|\frac{\partial w\pm}{\partial N}(x,y)\right| = \left|\frac{\partial w\pm}{\partial N}(x,-y)\right|$$

$$\left|\frac{\partial w\pm}{\partial T}(x,y)\right| = \left|\frac{\partial w\pm}{\partial T}(x,-y)\right|$$

$$\left[\frac{\partial w}{\partial N}(x,y)\frac{\partial w}{\partial T}(x,y)\right]^{\pm}=\left[\frac{\partial w}{\partial N}(x,y)\frac{\partial w}{\partial T}(x,y)\right]^{\pm}$$

5.the same properties as in 4 are also hold for u

6.$\langle AN,N\rangle|$[redacted]$_1 = \langle AN,N\rangle|_{l_2}$ $\langle AT,T\rangle|_{l_1} = \langle AT,T\rangle|_{l_2}$

$\langle AN,T\rangle|_{l_1} = \langle AN,T\rangle|_{l_2}$

where
$$A \overset{\text{def}}{=} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

and N and T denote repectively the tangential and normal vectors on the internal domain D,that is,

$$N = \begin{cases} (-\sin\theta_0, \cos\theta_0) \text{ on } l_1 \\ (-\sin\theta_0, -\cos\theta_0) \text{ on } l_2 \end{cases} \quad \text{and} \quad T = \begin{cases} (\cos\theta_0, \sin\theta_0) \text{ on } l_1 \\ (\cos\theta_0, -\sin\theta_0) \text{ on } l_2 \end{cases}$$

We recall the following important results done by [F,J,R](for C^1-domains), [V], and [D,K].

Let $L^p_0(\partial D) = \{ f \in L^p(\partial D) : \int_{\partial D} f d\sigma = 0\}$.

THEOREM [V],[D,K]. *Let* D be a bounded Lipschitz domain with connected boundary.

There is $\varepsilon = \varepsilon(D) > 0$ such that if $1 < p < 2+\varepsilon$, then

$$(i)\ S: L^p(\partial D) \to L^p_1(\partial D),$$

$$(ii)\ -\frac{1}{2} I + K^* : L^p(\partial D) \to L_0^p(\partial D), \text{ and}$$

$$(iii)\ \frac{1}{2} I + K^* : L^p(\partial D) \to L^p(\partial D)$$

are invertible.

(See sec 2 for the relevent definitions of S and K^*.)

2. INVERTIBILITY OF LAYER POTENTIAL OPERATOR

For a natational simplicity, we assume

$$r_0 = \frac{1}{4} \text{ and } 0 < \theta_0 \leq \frac{\pi}{4}$$

(We can apply exactly the same ideas in the general case) Let Ω denote the diamond shape domain:

$\Omega = \{(x,y): -x\tan\theta_0 < y < x\tan\theta_0$ and $(x-\cos\theta_0)\tan\theta_0 < y < -(x-\cos)\tan\theta_0\}$

S and $\widetilde{S}$ the single layer potentials

$$Sf(X) = \frac{1}{2\pi}\int_{\partial\Omega} \log|X - Q| f(Q)d\sigma,\ X\in R^2$$

$$\widetilde{S}f(X) = \int \widetilde{\Gamma}(X - Q)f(Q)d\sigma,\ X\in R^2$$

where $\widetilde{\Gamma}$ is the fundamental solution of operator $aD_x^2 + dD_y^2$ and for $P\in\partial\Omega$

$$K^*f(P) = \int_{\partial\Omega} \langle\nabla\Gamma(P-Q), N(P)\rangle f(Q)d\sigma$$

$$\widetilde{K}^*f(P) = \int_{\partial\Omega} \langle\nabla\widetilde{\Gamma}(P-Q), N(P)\rangle f(Q)d\sigma.$$

THEOREM 2. The mapping

$T: L^2(\partial\Omega) \to L^2(\partial\Omega)$ *defined by*

(2.1) $$T(f) = (\frac{1}{2}I + K^*)f - (-\frac{1}{2}I + \widetilde{K}^*)\widetilde{S}^{-1}Sf$$

is invertible.

LEMMA 3. *Given* $f \in L^2(\partial\Omega)$ the function

$$u \overset{def}{=} \begin{cases} u^+ \overset{def}{=} S(\widetilde{S}^{-1}Sf) & \text{in } \Omega \\ u^- \overset{def}{=} S(f) & \text{in } R^n\backslash\Omega \end{cases}$$

satisfies the estimates

(2.2) $\|(\nabla u^+)\|L^2(\partial\Omega) + \|(\nabla u^-)\|L^2(\partial\Omega)$
$\le C\{\|T(f)\|^2 L^2(\partial\Omega) + \|(\nabla u^+)\|L^2(\partial\Omega)\|T(f)\|L^2(\partial\Omega) + Z(f)\}$

where $Z(f_j) \to 0$ whenever $f_j \to 0$ weakly in $L^2(\partial D)$.

PROOF: It suffice to prove the estimates (2.2) for $f(x,y)=f(x,-y)$ and $f(x,y) = -f(x,-y)$. We first assume that $f(x,y)=f(x,-y)$. We will use two vector field

$$\vec{\alpha}(x,y)=(\alpha_1,\alpha_2)\varphi(\sqrt{x^2+y^2}) \text{ and } \vec{\beta} = (\beta_1,\beta_2)\varphi(\sqrt{x^2+y^2})$$

where $\varphi \in \vec{C}_0^\infty([0,1/4))$ is a positive function, 1 in the interval $[0,1/5]$, and $\alpha_1, \alpha_2, \beta_1$ and β_2 are constants. Recall the Rellich identites

$$div(\vec{\alpha}\langle\nabla u^-\cdot\nabla u^-\rangle) = 2div(\langle\vec{\alpha},\nabla u^-\rangle\nabla u^-) + O(|\nabla u^-|^2) \text{ in } R^2\backslash\Omega$$

and

$$div(\vec{\beta}\langle\nabla u^+\cdot\nabla u^+\rangle) = 2div(\langle\vec{\beta},\nabla u^+\rangle\nabla u^+) + O(|\nabla u^+|^2) \text{ in } \Omega.$$

Integrating the above identities over $R^2\backslash\Omega$ and Ω we get respectively

(2.3)
$$\int_{\delta\Omega} \langle\vec{a},N\rangle(\langle\nabla u^-\cdot T\rangle^2 - \langle\nabla u^-,N\rangle^2)d\sigma$$
$$= 2\int_{\delta\Omega} \langle\vec{a},N\rangle(\langle\nabla u^-\cdot T\rangle\langle\nabla u^-,T\rangle d\sigma + O(\|(\nabla u^+)\|^2_{L2(B1)})$$

and

(2.4)
$$\int_{\delta\Omega} \langle\vec{\beta},N\rangle(\langle\nabla u^+,T\rangle\langle\nabla u^+,T\rangle - \langle A\nabla u^+,N\rangle\langle\nabla u^+,N\rangle)d\sigma$$
$$= 2\int_{\delta\Omega} \langle\vec{\beta},T\rangle(\langle A\nabla u^+,N\rangle\langle\nabla u^+,T\rangle d\sigma + O(\|\nabla u\|^2_{L2(B1)})$$

Using the transmission conditions

$$\langle\nabla u^-\cdot T\rangle = \langle\nabla u^+,T\rangle \text{ on } \partial\Omega$$
$$\langle\nabla u^-\cdot N\rangle = \langle A\nabla u^+,N\rangle + T(f) \text{ on } \partial\Omega$$

and adding (2.3)and (2.4) we can get

(2.5)
$$\int_{11U12}\left[\begin{array}{l}\langle\nabla u^+,T\rangle^2\left\{\begin{array}{l}\langle\vec{\beta},N\rangle(AT,T\rangle - 2\langle\vec{\beta},T\rangle\langle AT,N\rangle \\ +(1-\langle AT-N\rangle 2)\langle\vec{a},N\rangle - 2\langle a,T\rangle\langle AT,N\rangle\end{array}\right\} \\ + \langle\nabla u^+,T\rangle^2(-\langle AN-N\rangle\langle\vec{\beta},N\rangle - \langle AN-N\rangle^2\langle\vec{a},N\rangle) \\ - 2\langle\nabla u^+,T\rangle\langle\nabla u^+,N\rangle\left\{\begin{array}{l}\langle AN-N\rangle\langle\vec{\beta},T\rangle+\langle AT-N\rangle\langle AN-N\rangle\langle\vec{a},N\rangle \\ +\langle AN-N\rangle\end{array}\right\}\end{array}\right]d\sigma$$

$$=O(\|T(f)\|^2_{L2(\partial\Omega)})+\|(\nabla u^+)\|L^2(\partial\Omega)\|T(f)\|^2_{L2(\partial\Omega)}+(\|\nabla u\|^2_{L2(B1)})$$

Since $u(x,y) = u(x,-y)$ for all $(x,y) \in R^2$, from the Symmetry properties mentioned above the left hand side integral of (2.5) can be written in the form

(2.6)
$$\int_{11} P(a_1,\beta_1)\langle\nabla u^+,T\rangle^2 + Q(a_1,\beta_1)\langle\nabla u^+,N\rangle^2 - 2R(a_1,\beta_1)\langle\nabla u^+,T\rangle\langle\nabla u^+,N\rangle d\sigma$$

where

(2.10) $\int |\nabla u|^2 d\sigma \leq C\{\|T(f)\|^2_{L2(\partial\Omega)})+\|(\nabla u^+)\|L^2(\partial\Omega)\|T(f)\|^2_{L2(\partial\Omega)}+(\|\nabla u\|^2_{L2(B_1)}\}.^{11}$

where the constant C depends on θ_0, $a-d$ and $ad-1$. We can simmilary treat the case where $f(x,y) = -f(x,-y)$ and the other three corners of the lozenge-shaped domain Ω to conclude the estimates (2.2) when $ad \neq 1$ and $a \neq d$.

When $ad = 1$ and $a \neq d$, from (2.8) we can see

$$\widetilde{(c_1 c_2)} = (c_1, c_2)\begin{bmatrix} -\dfrac{a\sin^2\theta_0 - d\cos^2\theta_0}{a\cos^2\theta_0 + d\sin^2\theta_0} \end{bmatrix}$$

$$= -(c_1, c_2)\frac{\langle AN,N\rangle}{\langle AT,T\rangle}$$

and therefore we have

$$(2.11)\quad \int_{\partial\Omega} \{\langle AT,T\rangle\langle\nabla u^+,T\rangle^2 - \langle AN,N\rangle\langle\nabla u^+,N\rangle^2\}d\sigma$$

$$O(\|T(f)\|^2_{L2(\delta\Omega)})+\|(\nabla u^+)\|L^2(\delta\Omega)\|T(f)\|^2_{L2(\delta\Omega)}+(\|\nabla u\|^2_{L2(B1)}).$$

Let us assume again $f(x,y) = f(x,-y)$. Applying the Rellich identity (2.4) with vector field $N_{12}\varphi$, we have from(2.11)

$$(2.12)\quad \int_{11}\langle A\nabla u^+,N\rangle\langle\nabla u^+,T\rangle d\sigma$$

$$=O(\|T(f)\|^2_{L2(\partial\Omega)})+\|(\nabla u^+)\|L^2(\partial\Omega)\|T(f)\|^2_{L2(\partial\Omega)}+(\|\nabla u\|^2_{L2(B1)}).$$

If we apply again Rellich formula (2.4) with vector field $e_2=(0,1)\psi$ over the domain $\Omega\cap R^2_+$, we have

$$\int_{\{(x,0):0\leq x\leq 1/4\}}\langle AT,T\rangle\langle\nabla u^+,T\rangle^2 - \langle AN,N\rangle\langle\nabla u^+,N\rangle^2 d\sigma$$

$$= \int_{11}\langle e_2,T\rangle\langle A\nabla u^+,N\rangle\langle\nabla u^+,T\rangle d\sigma$$

$$+ \int_{11}\langle e_2,T\rangle(\langle AT,T\rangle\langle\nabla u^+,N\rangle^2 - \langle AN,N\rangle\langle\nabla u^+,N\rangle^2)d\sigma$$

But from the symmetry property (3), $\langle\nabla u^+,N\rangle=0$, on $\{(x,0);x\in R\}$ this together with (2.11),(2,12), and(2,13) we have

$$(2.15)\quad \int_{\{(x,0):0\leq x\leq 1/4\}}|\nabla u|^2 d\sigma$$

$$=O(\|T(f)\|^2_{L2(\partial\Omega)})+\|(\nabla u^+)\|L^2(\partial\Omega)\|T(f)\|_{L2(\partial\Omega)}+\|\nabla u\|^2_{L2(B1)}).$$

From the same reason, applying the Rellich formula over the domain we conlude

(2.10) $\int |\nabla u|^2 d\sigma \le C\{\|T(f)\|^2_{L^2(\partial\Omega)} + \|(\nabla u^+)\|_{L^2(\partial\Omega)}\|T(f)\|^2_{L^2(\partial\Omega)} + (\|\nabla u\|^2_{L^2(B_1)})\}$.[11]

where the constant C depends on θ_0, a-d and ad-1. We can simmilary treat the case where $f(x,y) = -f(x,-y)$ and the other three corners of the lozenge-shaped domain Ω to conclude the estimates (2.2) when $ad \neq 1$ and $a \neq d$.

When $ad = 1$ and $a \neq d$, from (2.8) we can see

$$\widetilde{(c_1 c_2)} = (c_1, c_2)\left(\frac{-a\sin^2\theta_0 - d\cos^2\theta_0}{a\cos^2\theta_0 + d\sin^2\theta_0}\right)$$

$$= -(c_1, c_2)\frac{\langle AN, N\rangle}{\langle AT, T\rangle}$$

and therefore we have

$$(2.11) \quad \int_{\partial\Omega} \{\langle AT,T\rangle\langle\nabla u^+,T\rangle^2 - \langle AN,N\rangle\langle\nabla u^+,N\rangle^2\}d\sigma$$

$$O(\|T(f)\|^2_{L^2(\delta\Omega)} + \|(\nabla u^+)\|_{L^2(\delta\Omega)}\|T(f)\|^2_{L^2(\delta\Omega)} + (\|\nabla u\|^2_{L^2(B_1)})).$$

Let us assume again $f(x,y) = f(x,-y)$. Applying the Rellich identity (2.4) with vector field $N_{12}\varphi$, we have from(2.11)

$$(2.12) \quad \int_{11}\langle A\nabla u^+,N\rangle\langle\nabla u^+,T\rangle d\sigma$$

$$=O(\|T(f)\|^2_{L^2(\partial\Omega)} + \|(\nabla u^+)\|_{L^2(\partial\Omega)}\|T(f)\|^2_{L^2(\partial\Omega)} + (\|\nabla u\|^2_{L^2(B_1)})).$$

If we apply again Rellich formula (2.4) with vector field $e_2=(0,1)\psi$ over the domain $\Omega\cap R^2_+$, we have

$$\int_{\{(x,0):0\le x\le 1/4\}}\langle AT,T\rangle\langle\nabla u^+,T\rangle^2 - \langle AN,N\rangle\langle\nabla u^+,N\rangle^2 d\sigma$$

$$= \int_{11}\langle e_2,T\rangle\langle A\nabla u^+,N\rangle\langle\nabla u^+,T\rangle d\sigma$$

$$+ \int_{11}\langle e_2,T\rangle(\langle AT,T\rangle\langle\nabla u^+,N\rangle^2 - \langle AN,N\rangle\langle\nabla u^+,N\rangle^2)d\sigma$$

But from the symmetry property (3), $\langle\nabla u^+,N\rangle=0$, on $\{(x,0);x\in R\}$ this together with (2.11),(2,12), and(2,13) we have

$$(2.15) \quad \int_{\{(x,0):0\le x\le 1/4\}}|\nabla u|^2 d\sigma$$

$$=O(\|T(f)\|^2_{L^2(\partial\Omega)} + \|(\nabla u^+)\|_{L^2(\partial\Omega)}\|T(f)\|_{L^2(\partial\Omega)} + \|\nabla u\|^2_{L^2(B_1)}).$$

From the same reason, applying the Rellich formula over the domain we connlude

(2.15) $\int_{\{(x,0):0\le x\le 1/4\}}|\nabla u|^2 d\sigma$

$=O(\|T(f)\|^2_{L2(\partial\Omega)})+\|(\nabla u^+)\|L^2(\partial\Omega)\|T(f)\|_{L2(\partial\Omega)}+\|\nabla u\|^2_{L2(B1)}).$

We can choose $h\in L^2(l_3)$ such that

$(S_{13}+\tilde{S}_{13})h = 2Sl_3(T(f)X_{\{(r,\theta_0):0<r<1/3\}})$ on $l_3 \overset{def}{=} \{(s,\theta_0) : -\infty<s<\infty\}$
where

$$S_{13}h(X) \overset{def}{=} \int_{13}\Gamma(X,Q)h(Q)d\sigma$$

and

$$\tilde{S}_{13}h(X) \overset{def}{=} \int_{13}\tilde{\Gamma}(X,Q)h(Q)d\sigma$$

Then it easy to show that if

(2.161) $\quad \omega \overset{def}{=} \begin{cases} \mathcal{S}_{13}(h - T(f)X_{\{(r,\theta_0):0<r<1/3\}}) \text{ in } \Omega\cap R^2{}_+\cap B_{1/2} \\ \tilde{\mathcal{S}}_{13}(-h) \text{ in } (B_{1/2}\setminus\Omega\cap R^2{}_+) \end{cases}$

$W = u + v$ s a weak soution of (1.4) in $B_{1/3}\cap R^2{}_+$. Also the directional derivative

$$D\theta_0 W \overset{def}{=} <(\cos\theta_0, \sin\theta_0), \nabla W>$$

is a week soution of (1.4) in $B_{1/3}\cap R^2{}_+$. We have from the result of [C,F,M,S],

(2.17) $\quad \int_{11}|D\theta_0 W|^2 d\sigma \le C\int_{\{(x,0):-1/4<x<1/4\}}|(D\theta_0 W)^*|^2 d\sigma \le C\int_{\partial(B1/4\cap R2+)}|D\theta_0 W|^2 d\sigma.$

And from a well knowm result for positive subsolutions of elliptic equations,

(2.18) $\quad Sup\{|D\theta_0 W(1/4,\theta)|^2 : 1/2\theta_0<\theta<\pi-\theta_0\} \le C\int_{(B2/7\setminus B1/8)\cap R2+}|D\theta_0 W|^2 d\sigma$

We can also apply the same argument as above for u and v separately in the set$\{(1/4,\theta):0<\theta<1/2\theta_0$ or $\pi-\theta_0 < \theta \le \pi$. Together with (2.16) and (2.18) the right hand side of (2.17) can be controlled by

(2.19) $\quad \|T(f)\|^2_{L2(\delta\Omega)})+\|(\nabla u^+)\|L^2(\delta\Omega)\|T(f)\|_{L2(\delta\Omega)}+\|\nabla u\|^2_{L2(B1)}+\|\nabla W\|^2_{L2(B1/2)}$

We can get the estimate (2.2) the case where $f(x,y) = -f(x,-y)$ by using (2.11) and repeating the argument as in the case where $f(x,y) = f(x,-y)$.

PROOF OF THEOREM 2:Case 1 : $ad < 1$.

For $t\in[1/2\max(a,b),1)$, the operator

$$T_t : L^2(\partial\Omega) \to L^2(\partial\Omega) \text{ defined by}$$

$$T_t(f) = (1/2I + K^*) - t(-1/2I + \tilde{K}^*)S^{-1}\tilde{S}f$$

is the transmission operator in the Theorem 2 corresponding to the operators

$$ta \frac{\partial}{\partial x}\frac{\partial}{\partial x} + td \frac{\partial}{\partial x}\frac{\partial}{\partial x} \quad \text{and } \Delta$$

inside and outside of Ω respectively and therefore we have the estimates

(2.20) $\|f\|_{L2(\partial\Omega)} \le C\{\|T_t(f)\|_{L2\partial\Omega} + \|\nabla u\|_{L2(B1)}\}$

where the constant C is as in Lemma 3. Note that the above constant C dose not depend on t (in fact, from the estimates(2.10) C depends on θ_0, 1-ad,and $a-d$). From the deep result of [C,Mc,Me], we have

(2.11) $\|(T_t - T_s)f\| \le C|t-s|\|f\|_{L2(\partial\Omega)}$

where the constant C depends on A and the Lipschitz character of Ω. Let

$$\varepsilon = \{s \in 1/2\max(a,d),1]: T_3 \text{ is invetible on } L^2(\partial\Omega)\}$$

Since I - 1/2max(a,d)A is positive definite,from Theorem 2.2 in section 2 it follows that T_a is invertible on $L^2(\partial\Omega)$ where s=1/2mas(a,d). Hence ε is not empty.

T_s is one-to-one for $s \in$ [1/2max(a,d), 1]; sinec if $T_s(f) = 0$, u as in the Lemma 3 is a week solution of the equation:

(2.22) $$\frac{\partial}{\partial x}\left\{((ta-1)X\Omega + 1)\frac{\partial u}{\partial x}(x,y)\right\} + \frac{\partial}{\partial x}\left\{((ta-1)X\Omega + 1)\frac{\partial u}{\partial x}(x,y)\right\} = 0$$

in the entire R^2 and therefore as in section 2 we can conclude f = 0 in $\partial\Omega$.

It is easy to see from (2.21) that ε is open (see for example [G,T]). Now it remains to be shown that ε is closed. To do this assume $s_j \to s$ and $s_j \in \varepsilon$. For given $g \in L^2(\partial\Omega)$, we can find $f_i \in L^2(\partial\Omega)$ such that

$$T_{sj}(f_j) = g \text{ on } \partial\Omega$$

If $\|f_j\|_{L2(\partial\Omega)} \le c < \infty$, we may assume that $f_j \to f$ weakly in $L^2(\partial\Omega)$. It is easy to see that $T_s(f_j) \to T_s(f)$ weakly in $L^2(\partial\Omega)$. (2.21) implies that for all $\phi \in L^2(\partial\Omega)$

$$\int_\partial (T_s(f) - g)\,\phi d\sigma = \int_\partial (T_s(f) - T_s(f_j))\,\phi d\sigma + \int_\partial (T_s(f_j) - T_{sj}(f_j))\,\phi d\sigma \to 0 \text{ as } j \to \infty.$$

Hence $T_s(f) = g$ and T_s is onto.

If $\|f_j\|_{L^2(\partial\Omega)}$ are not bounde, we may assume that $\|f_j\|_{L^2(\partial\Omega)} =1$ and $T_{s_j}(f_j) \to 0$ strongly in $L^2(\partial\Omega)$. From (2.21), it is easy to see that $T_j(f_j) \to 0$ strongly in $L^2(\partial\Omega)$.

Since T_s is one to one, we may assume that $f_j \to 0$ weakly in $L^2(\partial\Omega)$. But

$$1 = \|f_j\|_{L^2(\partial\Omega)} \leq C\|T_s(f_j)\|_{L^2(\partial\Omega)} + z(f_j) \to 0 \quad \text{as } j \to 0$$

,a contradition. Therefore T_s is invertible and $\varepsilon = [1/2\max(a,d),1]$.

Case 2 : $ad > 1$.

Using a similar methods as in case 1 with now the operator

$$T_t(f) = t(1/2I + K^*) - (-1/2I + \widetilde{K^*}\ )\ \widetilde{S}^{-1}Sf$$

we can show that T_1 is invertible.

Case 3: $ad = 1$

Suppose that T and ε are as in the proof of case 1. From case 1, we know that $[1/2\max(a,d),1] \subset \varepsilon$ and T_1 is one to one. We also have the estimates (2.21) but we don't have uniform estimates (2.20) in this case. However the estimates (2.19) is enough to prove T_1 is invertible if we repeat the arguments as in the case 1.

PROOF OF THEOREM 1: We may assume $\Omega \subset B_{r_0}$. It is easy to see that

$$(\nabla u)^* \in L^2(\partial B_{3/2r_0}) \cap L^2(\partial\Omega \setminus B_{1/2r_0}).$$

Set $\Xi = (B_{3/2r_0} \setminus D) \cup \Omega$. We can find $h_1 \in L^2(\partial\Xi)$ such that

$$S_{\partial\Xi}(h_1)(P) \stackrel{def}{=} \int_{\partial\Xi} \Gamma(P-Q)h_1(Q)d\sigma = u(P) \quad P \in \partial\Xi.$$

We also can find $h_2 \in L^2(\partial\Xi)$ such that

$$h_2 = \langle \nabla S^- \partial h_1, N\rangle - \langle \nabla u^+, N\rangle \text{ on } \partial\Xi$$

Then

$$w_1 = \begin{cases} u & \text{in } \Omega \\ u + S\partial\Xi(h_2) & \text{in } \Xi \setminus \Omega \\ u + S\partial\Xi(h_2) + S\partial\Xi(h_1) & \text{in } R^2 \setminus \Xi \end{cases}$$

$$\Delta w_1 = 0 \quad \text{in } R^2 \setminus \Omega.$$

Now we can take $h_3 \in L^2(\partial\Omega)$ such that

$$S(h_3) = w_1^+ - w_1^- \text{ on } \partial\Omega$$

From Theorem 2 we can find $f \in L^2(\partial\Omega)$ such that

$T(f) = \langle A\nabla v_1^+, N\rangle - \langle\nabla(v_1 + S(h_3))^-, N\rangle$ on $\partial\Omega$.

It is then easy to see that

$$v_2 \overset{def}{=} \begin{cases} w_1 + \tilde{S}(\tilde{S}^{-1}Sf) & \text{in } \Omega \\ w_1 + S(f) + S(h_3) & \text{in } R^2 \backslash \Omega \end{cases}$$

is a week solution of the equation

$$\frac{\partial}{\partial x}\left[((a-1)\chi\Omega + 1)\frac{\partial}{\partial x}(x,y)\right] + \frac{\partial}{\partial x}\left[((d-1)\chi\Omega + 1)\frac{\partial}{\partial x}(x,y)\right] = 0$$

in the entire R^2 and therefore $w_2 = 0$ in R^2. Therefore we have a representation formula for u in B_{r_0} which directly given us estimate(1.5)

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