Polytopes 및 Triangulated Spaces에 關한 研究

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A Study on Polytopes and Triangulated Spaces

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要 約

本 論文에서는 polytope (多角形 혹은 多邊形과 같은 유우크리드 空間의 部分集合을 나타내는데 使用되어온 하나의 總稱用語) 에 關한 基本的 位相構造를 例證했으며, 몇개의 補助定理를 利用하여 다음의 事實을 밝혔다.

$|K|$ 와 $|L|$ 를 各各 三角形化 한 K 와 L 을 가지는 有限 polytopes 라 할때 f가 $|K|$ 에서 $|L|$ 에로의 한 連續寫像이고 K 가 Star-related to L relative to f 이면 다음 (1), (2), (3), 과 같이 되는 $|K|$ 에서 $|L|$ 에로의 寫像 s 가 存在한다.

(1) s: $K \to L$, simplicial mapping

(2) $\exists\,{}^{\forall}x \in |K|$, ${}_{\exists}v_j \in L$ 는 both $f(x)$ and $s(x) \in \dot{S}_t(v_j)$

(3) s is homotopic to f.

1. Preliminary

In the present paper, we would study the topological structure for the polytope which has become a generic term used to denote those subsets of a Euclidean space, such as polygons or polyhedra which are constructed with rectilinear elements.

Above all things, we introduce that some examples have been spaces which are homeomorphic to some polytope and refer here to such things as geometric complexes, star topologies, barycentric subdivision and the basic geometry of polytope. And then we are going to study simplicial mappings

2. Geometric complex and polytopes

Definition. 2. 1. A set $\{a_0, a_1, \cdots, a_k\}$ of vector in E^n is pointwise independent provided that the vectors a_1-a_0, a_2-a_0, $\cdots$, a_k-a_0 are linealy independent. Here, $\{v_1, v_2, \cdots, v_k\}$ ($\subset$ vector space, k: finite) is linearly independent iff $\sum_{i=1}^{k} f_i v_i=\overline{0}$, $f_i \in$ Field$\rightarrow f_i=0$ $(i=1, \cdots, k)$

Definition. 2. 2. Let $A=\{a_0, a_1, \cdots, a_k\}$ be a set of $k+1$ pointwise independent point in E^k. And we denote by $\{h\in H^k | h=\sum_{i=0}^{k} f_i a_i,\ \sum_{i=0}^{k} f_i=1,\ f_i\geqq 0,\ a_i\in A\}$.

Then we define S_k by geometric k-simplex in E^n determined by A.

〈**Example. 2. 1°**〉 (1) S_0 : a set of only one point, $\{a_0\}$
(2) S_1 : a closed line segment, a_0——a_1.
(3) S_2 : a closed triangular plane region.
(4) S_3 : a closed solid tetrahedron, so on.

Definition. 2. 3. $H^k(\subset E^k)$ is k-dimensional hyperplane iff there exists a_0, $\{a_1, a_2, \cdots, a_k\}$ of E^n such that $H^k=\{h | h=a_0+\sum_{i=1}^{k} t_i a_i,\ t_i\in E^1\}$. As it were, if $a_0=\overline{0}=(0, 0, \cdots, 0)$, then H^k is k-dimensional vector subspace of E^n.

And so $H^1\subset E^2$, then $H^1=\{(x, 0) | x\in E^1\}$ is one dimensional hyperplane for E^2.

We denote open geometric k-simplex in E^n (determined by A) by $S_k=\{h\in H^k | h=\sum_{i=0}^{k} f_i a_i,\ \sum_{i=0}^{k} f_i=1,\ f_i\gtrless 0,\ a_i\in A)$.

Lemma. 2. 4. Let $A=\{a_0, a_1, \cdots, a_k\}$ $(k\leqq n,\ A\subset E^n)$ be pointwise independent. Then there exists unique k-dimmensional hyperplane such that

(1) $A\subset H^k$

(2) $h\in H^k\Rightarrow h=a_0+\sum_{i=1}^{k} g_i\cdot(a_i-a_0)$ $(h\neq\overline{0}\Rightarrow g_i$: unique)

Proof : (1) Let H^k be $\{h | h=a_0+\sum_{i=1}^{k} g_i(a_i-a_0),\ g_i \in E^1\}$.

Then (i) H^k is k-dimensional hyperplane by Definition 2. 3.

(ii) $\exists^{\forall} a_j\in A$, $a_j=a_0+\sum_{i=1}^{k}\delta_{ij}(a_j-a_0)$ where δ_{ij} is Kronecker delta. Hence a_j belongs to H^k for all j.

Thus $A\subset H^k$.

(2) $\{h-a_0 \mid h-a_0=\sum_{i=1}^{k} g_i(a_i-a_0),\ h\in H^k\}$ is k-dimensional subspace of E^n with basis $\{a_i-a_0\}$. Hence g_i is unique for h $(\neq\overline{0})$. And then we must claim the uniqueness of H^k.

Suppose that we have the other k-dimensional hyperplane F^k such that

(1) $A\subset F^k$

(2) ${}^{\forall}p\in F^k \Rightarrow p=a_0+\sum_{i=1}^{k} g_i(a_i-a_0)$ (If $p\neq 0 \Rightarrow g_i$: unique). Then there exist $B=\{b_1, b_2, \cdots, b_k\}$ and b_o such that $\{b_1, b_2, \cdots, b_k\}$ is linearly independent and $F^k=\{p \mid p=b_0+\sum_{i=1}^{k} f_i b_i\}$.

Since $A\subset F^k$, there exists $f_{ij}(\in E^1)$ such that $a_j=b_0+\sum_{i=1}^{k} f_{ij}b_i$ $(j=0, 1, 2, \cdots, k)$ for every a_j.

Putting $a_0=b_0+\sum_{i=1}^{k} f_{i0}\, b_i$, we gain $\quad a_j-a_0=\sum_{i=1}^{k} (f_{ij}-f_{i0})\cdot b_i$.

Since both $B=\{b_1, b_2, \cdots, b_k\}$ and $\{a_j-a_0\}$ $(j=1, 2, \cdots k)$ are linearly independent, there exists unque $b_i=\sum_{j=1}^{k} g_{ij}\times(a_j-a_0)(i=1, 2, \cdots, k)$ and so,

$p=b_0+\sum_{i=1}^{k} f_i \left(\sum_{j=1}^{k} g_{ij}(a_j-a_0)\right)$ belongs to H^k

Hence $F^k\subset H^k$, similarly $H^k\subset F^k$. Thus $F^k=H^k$.

Lemma. 2.5. Let $\langle p_0 p_1 \cdots p_k\rangle$ be the geometric k-simplex determined by a set $A=\{p_0, p_1, \cdots, p_k\}$ of $k+1$ pointwise independent points of E^n.

Then $\langle p_0 p_1 \cdots p_k\rangle$ is the convex hull of the set A.

Proof : To show that $\langle p_0 p_1 \cdots p_k\rangle$ is convex, let $\sum_{i=0}^{k} x_i\, p_i$ and $\sum_{i=0}^{k} y_i\, p_i$ belong to $\langle p_0 p_1 \cdots p_k\rangle$ where $\sum_{i=0}^{k} x_i=1$ and $\sum_{i=0}^{k} y_i=1$. we would show that $\sum_{i=0}^{k} tx_i p_i+\sum_{i=0}^{k} (1-t)y_i\, p_i=\sum_{i=0}^{k} \{(tx_i+(1-t)y_i\}p_i$ belongs to $\langle p_0 \cdots p_k\rangle$. But $\sum_{i=0}^{k}\{(tx_i+(1-t)\ y_i)\}=t\ \sum_{i=0}^{k}x_i+(1-t)\ \sum_{i=0}^{k}y_i=t+(1-t)=1$.

Hence $\sum_{i=0}^{k}(tx_i+(1-t)y_i)\times p_i$ belongs to $\langle p_0 \cdots p_k\rangle$.

And then we verify that $\langle p_0 p_1 \cdots p_k\rangle$ is convex hull. Now, let B be a convex set containing A and $\sum_{i=0}^{k} x_i p_i$ belong to $\langle p_0 p_1 \cdots p_k\rangle$

To show $\sum_{i=0}^{k}x_i\ p_i\in B\left(\sum_{i=0}^{k} x_i=1\right)$, by induction,

1^{st}-step ; Suppose $k=0 \Rightarrow \langle p_0 \rangle = \{p_0\} = A$. $A \subset B$. Hence $p_0 \in B$

2^{nd}-step ; Suppose that it is true for $k-1$ the same as the above assumption.

3^{rd}-step ; Let $x_k = \sum_{i=0}^{k} x_i p_i$ be a point in $\langle p_0 \cdots p_k \rangle$.

Then if $x_k = 0 \Rightarrow \sum_{i=0}^{k} x_i p_i = \sum_{i=0}^{k-1} x_i p_k + x_k p_k = \sum_{i=0}^{k-1} x_i p_i$, that is, $\sum_{i=0}^{k-1} x_i p_i \in B$ by 2^{nd}-step. And then

if $x_k \neq 0$, $\sum_{i=0}^{k} x_i p_i = \sum_{i=0}^{k-1} x_i p_i + x_k p_i$

$$=(1-x_k) \sum_{i=0}^{k-1} \frac{x_i}{1-x_k} p_i + x_k p_k = (1-x_k) p_i + x_k p_k.$$

But then, $\sum_{i=0}^{k-1} \frac{x_i}{1-x_k} = \frac{(x_0 + \cdots + x_k) - x_k}{1-x_k} = \frac{1-x_k}{1-x_k} = 1$ and since

$\sum_{x=0}^{k-1} \frac{x_i}{1-x_k} p_i$ belongs to B. Hence $\sum_{i=0}^{k} x_i p_i$ belongs to B. Thus B is the convex hull.

So to speak, we obtain that for a given subset A of E^n, the convex hull of A is the intersection of all convex subsets containing A. And so the convex hull of any subset A of E^n is convex.

From now on, we introduce some notations and examples. we represent that two geo metric simplexes, S^m and S^n, $m \leqq n$, are properly joined, if either

(1) $s^m \cap s^n = \phi$ or

(2) $s^m \cap s^n = s^k (k \leqq n)$

where s^k is a subsimplex of both s^m and s^n. And we denote a geometric n-simplex s^n by

$$s^n = \langle p_0 \, p_1 \cdots p^n \rangle$$

we call $\langle p_{j0} \cdots p_{jk} \rangle$ to be k-face of s^n and then geometric complex K is expressed by a (countable) collection of faces of geometric simples properly joined such that for all $s^n \in K$, every s^k is a face of s^n and belongs to K.

⟨Example 2.2.°⟩ $\langle p_0 \cdots \hat{p}_j \cdots p_n \rangle$ is a face of $\langle p_0 p_1 \cdots p_n \rangle$, which is obtained by deleting only p_j. As it were, in simplex $s^2 = \langle p_0 \; p_1 \; p_2 \rangle$, we have $\langle p_0 \, p_1 \, p_2 \rangle = \langle p_0 p_2 \rangle$ which is a face of $\langle p_0 \, p_1 \, p_2 \rangle$ as the following figure 1.

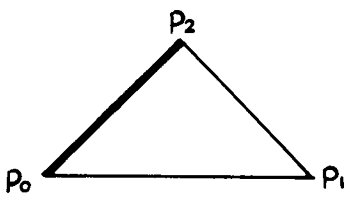

fig. 1.

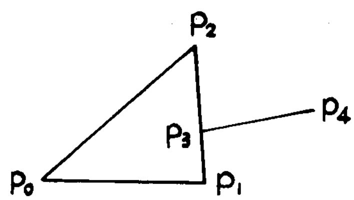

fig. 2.

⟨Example 2.3.°⟩ Let s^2 be $\langle p_0 \, p_1 \, p_2 \rangle$. Then $K = \{\langle p_0 \rangle$, $\langle p_1 \rangle$, $\langle p_2 \rangle$, $\langle p_0 \, p_1 \rangle$, $\langle p_1 \, p_2 \rangle$,

$\langle p_2p_3\rangle$, $\langle p_0p_1p_2\rangle\}$ is the geometric complex. Suppose that $K_0=\{\langle p_0\rangle, \langle p_1\rangle, \langle p_0p_1\rangle, \langle p_1p_2\rangle, \langle p_2p_0\rangle, \langle p_0p_1p_2\rangle, \langle p_3\rangle, \langle p_4\rangle, \langle p_3p_4\rangle\}$ as the above figure 2, then K_6 is the collection of simplexes but not geometric complex since $\langle p_1p_2\rangle \cap \langle p_3p_4\rangle=p_3$ is not a face of any simplex properly joined.

Definition 2.4. (1) K is defined the topological geometric complex by a (countable) collection of properly joined topological geometric simplex such that for all $s\in K$, every s' is a face of s and belongs to K. And we define a topology for a particular class of complex.

(2) we define $St(\sigma)=\{\sigma\in K|\sigma$: a face of simplex $s\}$ by the star of a simplex s and so we denote K(star-finite complex) by ${}^{\forall}\sigma\in K$, $St(\sigma)$ is finite.

(3) $\mathcal{T}$ is a star topology of a star-finite complex K if and only if $\mathcal{T}$ has a base β such that.

$$\beta=\{X|X\cap\sigma_i,\ i\text{ ; finite, }\sigma_i\text{ : simplexes } X\cap\sigma \text{ is open in } \sigma,\ \exists^{\forall}\sigma\in K\}$$

Remark : $|K|=\bigcup St(\sigma)=\bigcup\{\sigma|\sigma\in K$, K : star finite complex$\}$ is called the geometric carrier of the complex K. And let $(\bigcup\{\sigma|\sigma\in K\}, \mathcal{T})$ be a topological space and if the open star of a simplex σ $(\in K)$ is denote by $\dot{S}t(\sigma)$ such that $\dot{S}t(\sigma)$=the interior of carrier of $St(\sigma)$, that is, the collection of the open subset of $|K|$.

⟨Example 2.4.°⟩ Let $K=\{\langle p_0\rangle, \langle p_1\rangle, \langle p_2\rangle, \langle p_0p_1\rangle, \langle p_1p_2\rangle, \langle p_2p_0\rangle, \langle p_0p_1p_2\rangle\}$ be star finite complex of S^2.

Then $St(p_0)=\{\langle p_0\rangle, \langle p_0p_1\rangle, \langle p_0p_2\rangle, \langle p_0p_1p_2\rangle\}$ and each element of $St(p_0)$ is carrier and we suppose that topology for each element, we gain topological space $(\langle p_0\rangle, \mathcal{T}\langle p_0\rangle)$ which is carrier.

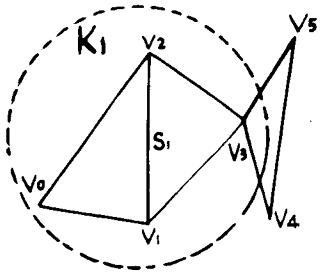

fig. 3

⟨Example 2.5°⟩ In the adjacent figure 3. $\dot{S}t(s_1)=K_1-\{\bigcup_3\}$
but

$St(s_1)=\{\langle v_1v_2\rangle, \langle v_0v_1v_2\rangle, \langle v_1v_2v_3\rangle\}$.

But we could obtain $\dot{S}t(\sigma)=St(\sigma)$ as the following example,

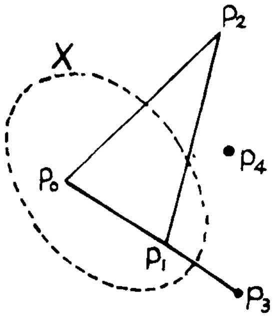

fig. 4

⟨Example2.6°⟩ In the adjacent figure 4, (that is, in topological space$(|K|\mathcal{T}_{|K|})$,

$\dot{S}t(\langle p_1\rangle)$ is the interior of the carrier of $St(\langle p_1\rangle)$

$$=\begin{Bmatrix}\langle p_0p_1p_2\rangle, & \langle p_1p_3\rangle\\ \langle p_1p_0\rangle, & \langle p_1p_2\rangle\end{Bmatrix}, i.e.,$$

$\dot{S}t(\langle p_1\rangle)=St(\langle p_1\rangle)$

3. Barycentric Subdivsion

Now we introduce a standard technique used for producing a triangulation of a given polytope such that the new triangulation is finer then the original. This subdivision is presented first for a complex consisting of a single simplex $s^n = \langle p_0 p_1 \cdots p_n \rangle$ together with all of its faces. Such a comlex is called the closure of a simplex and denote by $\mathrm{Cl}(s^n)$.

Definition 3.1. $\mathrm{cl}(s^n) = \{s \mid s : \text{face of } s^n = \langle p_0 p_1 \cdots p_n \rangle\}$ is called the closure of a simplex s^n. And $A = \{p_0,\ p_1, \cdots p_n\}$ are assumed to be pointwise independent and that

$$S_A^n = \{h \mid h = \sum_{i=0}^{n} x_i p_i,\ \sum_{i=0}^{n} x_i = 1,\ x_i \geqq 0\} = \langle p_0 p_1 \cdots p_n \rangle$$

is a geometric n-simplex and $S_A^n \subset E^n$.

The collection of all points $\dot{s}_j^k$, $k = 0,\ 1,\ 2,\ ,n$, where α_k is the number of K-simplexes in $\mathrm{cl}(s^n)$, will be the vertices of a new complex K^1, the first barycentric subdivision of $K = \mathrm{Cl}(s^n)$. Now returning to the vertices $\dot{s}_j^k$, we will take a subset of this points to be vertices of a simplex in K^1, $\langle \dot{s}_1 \dot{s}_2 \cdots \dot{s}_n \rangle$, iff $s_1 < s_2 < \cdots < s_i$ in K.

Lemma 3.2. If $\{\dot{s}_j^k : k = 0, 1, 2, \cdots, n,\ j = 1, 2, \cdots \alpha_k\}$, where α_k is the number of k-simpex in $\mathrm{Cl}(s^n)$, is the vertices of a new complex K^1, then the number of face for K^1 is $2^{n+1} - 1$.

Proof : $\dot{s}_1^0,\ \dot{s}_2^0, \cdots,\ \dot{s}_{n+1}^0$; ${}_{n+1}C_1$

$\dot{s}_1^1,\ \dot{s}_2^1, \cdots,\ \dot{s}_{n+1}^1$; ${}_{n+1}C_2$

....

....

$\dot{s}^n{}_{n1}$; ${}_{n+1}C_{n+1}$

Thus ${}_{n+1}C_1 + {}_{n+1}C_2 + \cdots + {}_{n+1}C_{n+1} = 2^{n+1} - 1$

Remark ; The point $\dot{s}^n$ is called the barycenter of the simplex s^n and in the centroid of the vertices p_i with epual weights assigned to each, $i, e.$, let $\dot{s}^k = \langle p_{i0} \cdots p_{ik} \rangle$ be a face $s^n = \langle p_0 p_1 \cdots p_n \rangle$, then $\dot{s}^k = \sum_{j=0}^{k} x_j p_{ij}$, $\sum_{i=0}^{k} x_j = 1$ and

$x_0 = \cdots = x_j = \dfrac{1}{k+1}$ weight $(j = 1,\ 2, \cdots, k)$.

〈**Example 3.1°**〉 Let s^2 be $\langle p_0 p_1 p_2 \rangle$. Then $\dot{s}^2 = \sum_{i=0}^{2} x_i p_i$, $\sum_{i=0}^{2} x = 1$, and $x_0 = x_1 = x_2$ like as the adjacent figure 5.

And so,

$K = \{s_0{}^0,\ s_1{}^0,\ s_2{}^0,\ s_1{}^1,\ s_2{}^1,\ s_3{}^1, s_2{}^2\}$
$= \{\langle p_0 \rangle,\ \langle p_0 p_1 \rangle,\ \langle p_1 \rangle,\ \langle p_1 p_2 \rangle,\ \langle p_2 \rangle,\ \langle p_2 p_0 \rangle,\ \langle p_0 p_1 p_2 \rangle\}$

$K^1 = \{\langle \dot{s}_0{}^0\ \dot{s}_1{}^1\ \dot{s}^2 \rangle\ \langle \dot{s}_1{}^1\ \dot{s}_1{}^0\ \dot{s}^2 \rangle \cdots \langle \dot{s}_0{}^0 \dot{s}_1{}^1 \rangle \cdots \langle \dot{s}_0{}^0 \rangle \cdots\}$

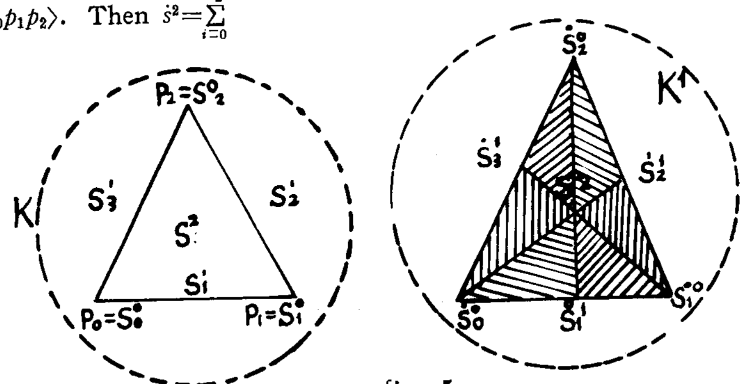

fig. 5

where K^1 is the 1^{st} barycentric subdivision of K.

As in the above fig. 5., this subdivision may now be done for each simplex of any geometric complex K and define a now complex K^1, the first barycentric subdivision of K.

It is evident that the geometric carrier of K and K^1 are identical.

Now we define the mesh of a geometric complex K by $\sup\{d | d$: the diameter of every simplex of $K\}$

Lemma 3.2. Let s^k be a geometric $k-$simplex.

Then $d(s^k)$ is equal to the length of its longest edge (or $1-$face).

Proof : Suppose $A=\{p_0p_1\cdots p_k\}$, the set of all vertices of s^k. $s^k=\langle p_0p_1\cdots p_k\rangle=\bigcap_{i=1}^{n} B_i$, where B_i contains A, is convexhull of A by Lemma 2.5. $d(s^k)=d(A)=d(\overline{a}, \overline{b})$ where a, b are endpoint respectively of $1-$face since $d(A)=d\left(\bigcap_{i=1}^{n} B_i\right)$ $(A\subset B_i,\ \forall_i)$.

Thus $d(A)$ is equal to $1-$face of the simplex

Theorem3.3. Let K be a geometric complex, finite dimension, n and finite mesh. λ and let m be the mesh of its 1^{st} barycentric subdivision K^1.

Then $m\ (K^1)\leqq\frac{n}{n+1}\lambda$.

Proof : Let an arbitrary s^k belong to K, $s^k=\langle p_{j1},\cdots p_{jk}\rangle$. Then the barycenter $\dot{s}^k$ of s^k equals to $\frac{1}{k+1}(p_{j1}+\cdots+p_{jk})$, where $\frac{1}{k+1}$ is the barycentric coordinates of $\dot{s}^k$ since $\sum_{i=0}^{k} f_i=1$, $f_0+\cdots+f_k=1$ from $f_0=\cdots=f_k$, thus $f_i=\frac{1}{k+1}$.

Now then, let $\langle\dot{s}^i\dot{s}^k\rangle$ be an arbitrary $1-$simplex of K^1 in the subdivision of s^k and s^i be a face of s^k in K as $s^i<s^k$

Put μ=the length of $\langle\dot{s}^i\dot{s}^k\rangle$. If s^i is $\langle p_0\cdots p_i\rangle$ then $s^{k-i-1}=\langle p_{i+1}\cdots p_k\rangle$ and if $\dot{s}^i\ \dot{s}^{k-i-1}$ is the line segment from $\dot{s}^i$ to $\dot{s}^{k-i-1}$, then$\langle\dot{s}^i\dot{s}^{k-i-1}\rangle\supset\langle\dot{s}^i\dot{s}^k\rangle$.

And so,

the weight of the barycenter $\dot{s}^i=\frac{1}{k+1}[p_0+\cdots+p_i]=\frac{i+1}{k+1}$

while $\dot{s}^{k-i-1}$ has weight $\frac{k-i}{k+1}$ as centroids of the vertices, since $\dot{s}^{k-i-1}$ has $[(k-i-1)+1]$ of the vertices and its weight is $\frac{1}{k+1}\times(k-i)$.

In the result, $\dot{s}^k=\frac{i+1}{k+1}\dot{s}^i+\frac{k-i}{k+1}\dot{s}^{k-i-1}$

$$=\frac{1}{k+1}[p_0+\cdots+p_k],$$

where $\dot{s}^k$ is the centroid of these two perticles as the below figure6.

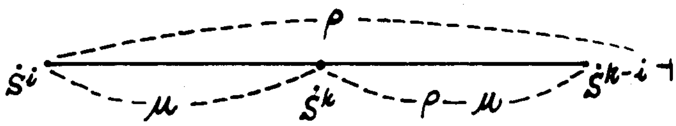

fig. 6.

Putting the length of line segment from $\dot{s}^i$ to $\dot{s}^{k-i-1}$ by $l\,\overline{(\dot{s}^i\,\dot{s}^{k-i-1})}=\rho$, then we obtain

$$\left(\frac{i+1}{k+1}\right)\mu=\left(\frac{k-i}{k+1}\right)(\rho-\mu)$$

or $$\left(\frac{i+1}{k+1}+\frac{k-i}{k+1}\right)\mu=\frac{k-i}{k+1}\rho.\ \ i.e.,\ \ \mu=\frac{k-i}{k+1}\rho$$

since ρ does not exceed the diameter of s^k, we conclude that $\mu\leqq\frac{k-i}{k+1}\lambda\leqq\frac{k}{k+1}\lambda\leqq\frac{n}{n+1}\lambda$

thus $m(K^1)\leqq\frac{n}{n+1}\lambda$

Corollary 3.3. If $K^{(k)}$ is the k^{th}-barycentric subdivision of K, K is n—dimensional geometric complex and the mesh of K is λ ($<\infty$), then $m(K^{(k)})$ converges to zero as $k\to\infty$.

4. Simplicial mappings

Definition 4.1. Let $|K|$ and $|L|$ be polytopes with triangulations K and L respectively.

Putting $K_v=\{p_i|p_i$: vertex of $K\}$ and

$L_v=\{\sigma_i|\sigma_i$: vertex of $L\}$.

we define transformation f from K_v into L_v(f : a possibly many-to-one), satisfying conditions that if $\langle p_0\cdots p_n\rangle$ is a simplex of K, then $f(p_0)\cdots f(p_n)$ (not necessarily distinct) are the vertices of a simplex of L and that the function f is a continuous extension if and only if $f(x)=f\left(\sum_{i=0}^n x_i\,p_i\right)=\sum_{i=0}^n x_i f(p_i)$ such that $x=\sum_{i=0}^n x_i p_i$, $\sum_{i=0}^n x_i=1$, $x_i\geqq 0$ if $s^n=\langle p_0\cdots p_n\rangle$ is a simplex of K and $x\in s^n$.

The above mapping f is called a simplicial mapping.

Lamma 4.2. A simplicial mapping f is continuous.

Proof ; Let x belong to $|K|$. Then there exists the x in $s^k=\langle p_0p_1\cdots p_k\rangle$ which is belonged to K, and so,

$f(x+\Delta x)-f(x)=\sum_{i=0}^k(x_i+\Delta x_i)f(p_i)-\sum_{i=0}^k x_i f(p_i)=\sum_{i=0}^k \Delta x_i\,f(p_i)$. Hence $\sum_{i=0}^k \Delta x_i\,f(p_i)$ converges to zero when Δx_i approaches to zero.

From the above definition 4.1., we note that the mapping f from K_v into L_v, such that $\langle p_0 \cdots p_n \rangle \in K$, $f(p_0), \cdots, f(p_n)$ are vertices of a simplex of L, is a simplicial mapping.

Lemma 4.3. In a Euclidean space E^k, let $\{p_n\}$ and $\{q_n\}$ be two sequences in polytope $|K|$ $(\subset E^k)$ such that $\lim_{n\to\infty} p_n = p$, $\lim_{n\to\infty} q_n = q$ respectively, and let $\overline{p_n q_n}$ be the length of $[p_n, q_n]$ which is the line segment from p_n to q_n. If x_n belongs to $[p_n, q_n]$ for each n, and if there exists $\lim_{n\to\infty} d(x_n, p_n)$ as $n\to\infty$, then there exists a x on $[p, q]$ such that

(1) $\lim_{n\to\infty} d(x_n, p_n) = d(x, p)$ and (2) $\lim_{n\to\infty} x_n = x$.

Proof ; Put $p_n = (p_n^1, p_n^2, \cdots p_n^k)$, $q_n = (q_n^1, \cdots, q_n^k)$ and $x_n = (x_n^1, \cdots, x_n^k)$, $x = (x^1, x^2, \cdots, x^k)$.

Then $d(x_n, p_n) = \sqrt{\sum_{i=1}^{k} (x_n^i - p_n^i)^2}$ and $x_n = (1-t_n)p_n + t_n q_n$ $(0 \leqq t_n \leqq 1)$

Letting $t_n = \frac{1}{2} - \frac{1}{n+1}$, we obtain $\lim_{n\to\infty} x_n = \frac{1}{2}p + \frac{1}{2}q = x$. *i. e.*, $x \in [p, q]$, since $x_n = \left(\frac{1}{2} + \frac{1}{n+1} p_n\right) + \left(\frac{1}{2} - \frac{1}{n+1}\right) q_n$ and $\lim_{n\to\infty} d(x_n, p_n) = \sqrt{\sum_{i=1}^{k} \left[\lim_{n\to\infty} (x_n^i - p_n^i)\right]^2}$ ①. While $\lim_{n\to\infty} x_n = \lim_{n\to\infty} (x_n^1, \cdots, x_n^k) = (x^1, x^2, \cdots, x^k) = x$ and $\lim_{n\to\infty} p_n = \lim_{n\to\infty} (p_n^1, p_n^2, \cdots, p_n^k) = (p^1, \cdots, p^k) = p$

The formula ① becomes $\sqrt{\sum_{i=1}^{k} (x^i - p^i)^2} = d(x, p)$.

To show the existence of $\lim_{n\to\infty} d(x_n, p_n)$, we verify that there exists $t_n = \frac{1}{2} - \frac{1}{n+1}$.
Now, assume $(x_n - p_n) = t_n(q_n - p_n)$. $d(x_n, p_n) = |t_n| d(q_n, p_n)$ approaches to $|t_n| d(q, p)$ $(n \to \infty)$, while x_n converges to x as p_n, q_n and t_n approach to p, q and t respectively.

Hence there exists $d(x, p)$ provided that we take t_n by $t_n = \frac{1}{2} - \frac{1}{n+1}$

The proof is complete.

Lemma 4.4. If $v_0, v_1, \cdots, v_k$ are the vertices of a star finite complex and $\dot{S}t(v_i)$ is the open stars of $v_i (i=0, 1, \cdots, k)$, then $\langle v_0 v_1 \cdots v_k \rangle \in K$ if and only if $\bigcap_{i=0}^{k} \dot{S}t(v_i) \neq \phi$.

Proof : $(\rightarrow)$ ᄀV $\langle v_0 v_1 \cdots v_k \rangle \in K$, let $h = \sum_{i=0}^{k} x_i v_i$, $\sum_{i=0}^{k} x_i = 1$, $x_i > 0$, then there exists $h \in \dot{S}t(v_i)$.

thus $\bigcap_{i=0}^{k} \dot{S}t(v_i) \neq \phi$.

$(\leftarrow)$ Let $v_0, \cdots, v_k$ be any vertices, we can represent $v_0, \cdots, v_k$ as positive. Hence $\langle v_0 v_1 \cdots v_k \rangle$ forms a simplex of K.

From the above lemmas, we note that if K and L are triangulations of the polytopes $|K|$ and $|L|$ respectively, and if f is a continuous mapping of $|K|$ into $|L|$, then K is called star-related to L relative to f. And for every p_i of K, there is a vertex v_j of L such that $f(\dot{S}t(p_i)) \subset \dot{S}t(v_j)$ where p_i and v_j belong to K, L respetively.

Theorem 4.5. Let both $|K|$ and $|L|$ be finite polytopes with triangulations K and L respectively, and if f is a continuous mapping of $|K|$ into $|L|$.

If K is star-related to L relative to f, then there exists a mapping S of $|K|$ into $|L|$ such that

(1) $s: K \to L$, simplicial mapping

(2) $^{\forall}x \in |K|$, $\exists^{v_j} \in L$ such that both $f(x)$ and $s(x)$ belong to $\dot{S}t(v_j)$.

(3) s : homotopic to f.

Proof ; Assume $K_v = \{v|v:$ vertex of $K\}$ and $L_v = \{v|v:$ vertex of $L\}$, For all p_i of K_v, there is $v_j(i)$ of L_v such that $f(\dot{S}t(p_i)) \subset \dot{S}t(v_j(i))$. And then we define a correspondence s between the vertices of K and those of L by setting

$$s(p_i) = v_j(i).$$

Then for all p_i of K_v, there exists $v_j(i) = s(p_i)$ of L and for all $\langle p_0 p_1 \cdots p_k \rangle$ of K, $\bigcap_{i=0}^{k} \dot{S}t(p_i)$ is not empty by Lemma 4.4.

Since $f\left(\bigcap_{i=0}^{k} \dot{S}t(p_i)\right) \subset \bigcap_{i=0}^{k} f\left(\dot{S}t(p_i)\right) \subset \bigcap_{i=0}^{k} \dot{S}t(v_{j(i)})\Big) = \bigcap_{i=0}^{k} \dot{S}t(s(p_i))$ is not empty.

Again from Lemma 4.4. $\langle s(p_0) \cdots s(p_k) \rangle$ is a simplex of L. Thus s is simplicial, and by barrycentric extension we obtain a continuous mapping of $|K|$ into $|L|$.

To show that (2) succeed, every point x in $|K|$ lies in the interior of some simplex s^k of K, s^k taken to be of minimum dimension, *i, e.*, $s^k = \langle p_0 p_1 \cdots p_k \rangle \Rightarrow x = \sum_{i=0}^{k} x_i p_i$, $\sum_{i=0}^{k} x_i = 1$, $x_i > 0$.

If p is any vertex of s^k, then x lies in $\dot{S}t(p)$. By definition 4.1, $f(x) \in f(\dot{S}t(p)) \subset \dot{S}t(s(p))$ *i.e.* $f(x) \in \dot{S}t(s(p))$. But also $s(x)$ lies in $\dot{S}t(s(p))$, since $s(x) = \sum_{i=0}^{k} x_i\, s(p_i)$, $\sum_{i=0}^{k} x_i = 1$, and $x_i > 0$.

Thus the mapping satisfies condition (2).

It remains to show that s is homotopic to f. we define h by $|K| \times I^1 \to |L|$ such that $h(x, 0) = f(x)$ and $h(x, 1) = s(x)$, for each x of $|K|$. Let x be a point of $s^k = \langle p_0 \cdots p_k \rangle$ in K. Since s is simplicial, $s(p_0), \cdots, s(p_k)$ are vertices of a simplex s^* in L.

Having that $f(x)$ lies in $\dot{S}t(s(p_i))$ for each vector of s^k, it follows that $f(x)$ is a point of $\bigcap_{i=0}^{k} \dot{S}t(s(p_i))$, which is precisely the simplex s^* of L. Having both $f(x)$ and $s(x)$ in the same simplex s^* of L, we make use of the convexity of s^* and join $f(x)$ to $s(x)$ by a (unique) line segment in s^*.

Properly metrized, this line segment will be the image under a homotopy h of the line segment $x \times I^1$ in the homotopy cylinder $|K| \times I^1$, Letting $d(f(x), s(x))=1$, we write in vector notation,

$h(x, t)=(1-t)f(x)+ts(x)$

The continuity of h as defined here is a consequence of Lemma 4.3.

5. Conclusion

We have looked at a special class of continuous mappings of one polytope into another, namely, those mappings which carry simplexes linearly onto simplexes.

With the aid of Lemma 4.2, 4.3 and 4.4, we have completed the proof of theorem 4.5.

From the present proccess, we see that it is shown the simplicial approximation theorem [5] if we next replace the triangulations K and L by barycentric subdivisions K^* and L^*, L^* being chosen to yield the desired accuracy of approximation and K^* being chosen so as to be star related to L^* relative to f.

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