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# OPTIMALITY AND DUALITY FOR MULTIPLE OBJECTIVE NONCONVEX OPTIMIZATION PROBLEMS

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Optimality theorems of Fritz John type are established for multiple objective nonconvex optimization problems. Dual problems are given for these problems and it is shown that duality theorems hold without any form of contsraint qualifications.



## 1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we use the following conventions for vectors in the n-dimensinal Euclidean space $R^n$;

$x < y$ if and only if $x_i < y_i$, $i = 1,2,\dots,n$,
$x \leq y$ if and only if $x_i \leq y_i$, $i = 1,2,\dots,n$,
$x \leq y$ if and only if $x \leq y$ and $x \neq y$,
$x < y$ is the negation of $x < y$,
$x \leq y$ is the negation of $x \leq y$.

We consider the following multiple objective optimization problems:

(P) minimize $f(x)$ subject to $g(x) \leq 0$,
(PE) minimize $f(x)$ subject to $g(x) \leq 0$, $h(x) = 0$.

The functions $f:R^n \to R^p$, $g:R^n \to R^m$ and $h:R^n \to R^k$ are assumed to be differentiable. We are mainly concerned with weakly efficient solutions for (P) and efficient solutions for (PE).

**DEFINITION 1.1** x is said to be an efficient solution (weakly efficient solution) for (P) if  is a (P)-feasible solution and for any (P)-feasible

solution x, $f(x) \leq f(x)$ $(f(x) \leq f(x))$.
Similarly, we can define efficient solutions and weakly efficient solutions for (PE).

Recently, Singh [8] studied Kuhn-Tucker type optimality criteria for (PE) under generalized convex assumptions. The first aim of this paper is to establish Fritz John type optimality criteria for (PE) under generalized convex assumptions.

Mond and Weir [7] suggested various dual problems for single objective(i.e., scalar) optimization problems which are different from Wolfe [12] dual problems, and roved that duality theorems hold, under generalized convex assumption, between primal problems and dual problems. Recently, Weir [10], Egudo [5], and Weir and Mond [11] considered Mond-Weir [7] type dual problems for (P) and (PE), and obtained some duality results which are based on the proper efficiency, the efficiency and the weak efficiency respectively. In this paper, we consider the following Mond-Weir [7] type dual problems for (P) and (PE) which are slightly different from dual problems of the three authors mentioned above.

(D) maximize $f(v)$ subject to

(1) $\nabla\mu^t f(v) + \nabla y^t g(v) = 0,$

(2) $y_i g_i(v) \geq 0,\ i = 1,2,\ldots,m,$

(3) $(\mu,y) \geq 0.$

(DE) maximize $f(v)$ subject to

(4) $\nabla\mu^t f(v) + \nabla y^t g(v) + \nabla z^t h(v) = 0,$

(5) $y^t g(v) \geq 0,$

(6) $z^t h(v) \geq 0,$

(7) $(\mu,y) \geq 0,\ (\mu,y,z) \neq 0.$

By the similar method to Definiton 1.1, we can define efficient solutions and weakly efficient solutions for (D) and (DE).
Our dual problems, in the case of scalar optimization problems, are reduced to problems which are very similar to those of Bector, Chandra and Bector [3]. The second aim of this paper is to obtain duality theorems which hold, without any form of constraint qualifications and under generalized convex assumptions, between (P) and (D), and between (PE) and (DE).

## 2. OPTIMALITY CRITERIA

Now, we study in this section Fritz John type (necessary/sufficient) optimality theorems for (P) and (PE).

**THEOREM 2.1** ([4],[11]). If x is a weakly efficient solution for (P), then there exist $\mu \in R^p$ and $y \in R^m$ such that

$$\nabla \mu^t f(x) + \nabla y^t g(x) = 0,$$
$$y^t g(x) = 0,$$
$$(\mu, y) \geq 0.$$

In the same method as the proof of Theorem 11.3.1 in [6], we can prove the following theorem:

**THEOREM 2.2.** Suppose that the constraint function h is continuously differentiable at $x \in R^n$. If x is a weakly efficient solution solution for (PE), then there exist $\mu \in R^p$, $y \in R^m$ and $z \in R^k$ such that

$$\nabla \mu^t f(x) + \nabla y^t g(x) + \nabla z^t h(x) = 0,$$
$$y^t g(x) = 0,$$
$$(\mu, y) \geq 0. \; (\mu, y, z) \neq 0.$$

We now prove the following Fritz John type sufficient optimality theorems for (PE);

**THEOREM 2.3.** Let $\mu \in R^p$, $y \in R^m$ and $z \in R^k$ and $x \in R^n$, along with $\mu$, y and z, satisfy the following conditions:

(8) $\nabla \mu^t f(x) + \nabla y^t g(x) + \nabla z^t h(x) = 0,$

(9) $y^t g(x) = 0,$

(10) $g(x) \leq 0,$

(11) $h(x) = 0,$

(12) $(\mu, y, z) \geq 0.$

If f is pseudoconvex at x, and $g_I$ and h are strictly pseudoconvex at x, where $I = \{i : g_I(x) = 0\}$, then x is a weakly efficient solution for (PE).

**proof.** Suppose that x is not a weakly efficient solution for (PE). Then there exists $x^* \in R^n$ such that $f(x^*) < f(x)$, $g(x^*) \leq 0$ and $h(x^*) = 0$. since $f(x^*) < f(x)$, and f is pseudoconvex at x, we have

(13) $(x^* - x)^t \nabla f(x) < 0.$

Since $g_I(x^*) \leq g_I(x)$ and $g_I$ is strictly pseudoconvex at x, we have

(14) $\quad (x^* - x)^t \nabla g_I(x) < 0.$

Since $h(x^*) \leq h(x)$ and h is strictly pseudoconvex at x, we have

(15) $\quad (x^* - x)^t \nabla h(x) < 0.$

Let $J = \{ i : g_I(x) < 0\}$. Then, from (9) and (10), $I \cup J = \{1,2,\ldots,m\}$

and for all $j \in J$, $y_j = 0$. Hence, form (12), (13), (14) and (15), we have

$$(x^* - x)[\nabla \mu^t f(x) + \nabla y^t g(x) + \nabla z^t h(x)] < 0,$$

which contradicts (8). Hence the result holds.

**REMARK 2.1.** Theorem 2.3 is a generalization of results in [1] and [9].

**THEOREM 2.4.** Let $\mu \in R^p$, $y \in R^m$, $z \in R^k$ and $\dot{x} \in R^n$, along with and z, satisfy the following conditions;

(16) $\quad \nabla \mu^t f(x) + \nabla y^t g(x) + \nabla z^t h(x) = 0,$

(17) $\quad y^t g(x) = 0,$

(18) $\quad g(x) \leq 0,$

(19) $\quad h(x) = 0,$

(20) $\quad (\mu,y,z) \geq 0.\ (\mu,y,z) \neq 0.$

Assume that

(a) f is quasiconvex at $\bar{x}$ and $\bar{y}^t g + \bar{z}^t h$ is strictly pseudoconvex at $\bar{x}$; or

(b) $\mu^t f$ is quasiconvex at x and $y^t g + z^t h$ is strictly pseudoconvex at x; or

(c) f is quasiconvex at x, $y^t g$ is strictly pseudoconvex at x and $z^t h$ is quasiconvex at x; or
(d) $\mu^t f$ is quasiconvex at x, $y^t g$ is strictly pseudoconvex at x and $z^t h$ is quasiconvex at x,

then x is an efficient solution (weakly efficient solution) for (PE).
**proof.** (a) Suppose that x is not an efficient solution for (PE). Then there exists $x^* \in R^n$ such that $f(x^*) \leq f(x)$, $g(x^*) \leq 0$ and $h(x^*) = 0$. By the quasiconvexity of f at x, we have $(x^* - x)^t \nabla f(x) \leq 0$.Thus $\mu \geq 0$ implies that

$$(x^* - x)^t \nabla \mu^t f(x) \leq 0.$$

Therefore, from (16), we have

$$(x^* - x)^t [\nabla y^t g(x) + \nabla z^t h(x)] \geq 0.$$

Thus, by the strict pseudoconvexity of yg + zh at x, we have

$$y^t g(x^*) + z^t h(x^*) > y^t g(x) + z^t h(x).$$

Since $h(x*) = 0$, from (19), we have $y^t g(x^*) > y^t g(x)$. From (17), we have

$$y^t g(x^*) > 0.$$

which contradicts the fact that $y^t g(x^*) \leq 0$. Hence the result holds.

(b) Suppose that x is not an efficient solution for (PE). Then there exists $x* \in R^n$ such that $f(x^*) \leq f(x)$, $g(x^*) \leq 0$ and $h(x^*) = 0$. Since $\mu \geq 0$ and $f(x^*) \leq f(x)$ we have $\mu^t f(x^*) \leq \mu^t f(x)$. Thus, by the quasiconvixity $\mu^t f$ at x, we have

$$(x^* - x)^t \nabla \mu^t f(x) \leq 0.$$

By the same method as the proof of the part (a), we can prove he part (b).

(c) Suppose thar x is not an efficient solution for (PE). Then there exists $x* \in R^n$ such that $f(x^*) \leq f(x)$, $g(x^*) \leq 0$ and $h(x^*) = 0$. Then, by the quasiconvexity of f at x, we have $(x^* - x)^t \nabla f(x) \leq 0$. Since $\mu \geq 0$, we have

(21) $(x^* - x)^t \nabla \mu^t f(x) \leq 0.$

Since $z^t h(x^*) = z^t h(x)$ and $z^t h$ is quasivonbex at x, we have

(22) $(x^* - x)^t \nabla z^t h(x) \leq 0.$

From (16), (21) and (22), 23 have

$$(x^* - x)^t \nabla z^t g(x) \geq 0.$$

Thus, by the strict pseudoconvexity of $y^t g$ at x, we have $y^t g(x^*) > y^t g(x)$. From (17) we have

$$y^t g(x^*) > 0,$$

which contradicts the fact that $yg(x^*) \leq 0$. Hence the result holds.

(d) Suppose that x is not an efficient solution for (PE). Then there exists $x^* \in R^n$ such that $f(x^*) \leq f(x)$, $g(x^*) \leq 0$ and $h(x^*) = 0$. since $\mu \geq 0$ and $f(x^*) \leq f(x)$, we have $\mu^t f(x^*) \leq \mu^t f(x)$. Thus, by the quasiconvexity of $\mu^t f$ at x, we have

$$(x^* - x)^t \nabla \mu^t f(x) \leq 0.$$

By the same method as the proof of the part (c), we can prove the part (d).

**REMARK 2.2.** Bector and Bector [2] proved a Kuhn-Tucker sufficient optimality theorem for the scalar minimization problem under the assumption (a) in Theorem 2.4.

## 3. DUALITY THEOREMS

We now prove the following weak duality and strong duality relating (P) and (D) :

**THEOREM 3.1.** If, for any (P)-feasible solution x and any (D) feasible solution $(v,\mu,y)$, f is pseudovconvex at v and g is strictly pseudoconvex at v, then $f(x) < f(v)$.

**Proof.** Suppose that there exist a (P) - feasible solution x and a (D) -feasible solution $(v,\mu,y)$ such that $f(x) < f(v)$. Then, the pseudoconvexity of f at v implies that

(23) $(x - v)^t\nabla f(v) < 0.$

**Case 1:** $y=0$.

From (3), we have $\mu \geq 0$. Thus (23) implies that $(x - v)^t\nabla\mu^t f(v) < 0$. this contradicts (1).

**Case 2:** $y \neq 0$.

Let $M = \{ i: y_i > 0\}$ and $i \in M$. From (2) , we have $g_i(v) \geq 0$. Since, $g_i(x) \leq 0$, we have $g_i(x) \leq g_i(v)$. Thus, the strict pseudoconvexity of at $g_i$ at v implies that $(x - v)^t\nabla g_i(v) < 0$. Consequently, we have

(24) $(x - v)^t\nabla y^t g(v) < 0.$

On the other hand, (23) implies that

(25) $(x - v)^t\nabla\mu^t f(v) \leq 0.$

From (24) and (25), we have

$$(x - v)^t[\nabla\mu^t f(v) + \nabla y^t g(v)] < 0,$$

which contradicts (1). Hence the result follows.

**THEOREM 3.2.** Let x be a weakly efficient solution for (P). Then there exist $\mu \in R^p$ and $y \in R^m$ such that $(v,\mu,y)$ is a (D) -feasible soluion and the objective values of (P) and (D) are equal. Moreover, if , for any (P)-feasible solution x and any (D)-feasible solution $(v,\mu,y)$, f is pseudoconvex at v and g is strictly pseudoconvex at v, then $(v,\mu,y)$ is a weakly efficient solution for (D).

**Proof.** By Theorem 2.1, there exist $\mu \in R^p$ and $y \in R^m$ such $\nabla\mu^t f(x) + \nabla y^t g(x) = 0$, $y^t g(x) = 0$ and $(\mu,y) \geq 0$. Since, $y \geq 0$, $g(x) \leq 0$ and $y^t g(x) = 0$, we have $y_i g_i(x) = 0$, $i=1,2,\ldots\ldots, m$, Thus $(v,\mu,y)$ is a (D)-feasidble solution and clearly the objective values of (P) and (D) are equal.

By Theorem 3.1, $f(x) < f(v)$ for any (D)-feasible solution $(v,\mu,y)$. Since $(v,\mu,y)$ is a (D)-feasible solution, $(v,\mu,y)$ is a weakly efficient solution for (D). Hence the result holds.

Now, we prove the following weak duality and strong duality for (PE) and (DE):

**THEOREM 3.3.** If, for any (PE)-feasible solution x and any (DE)-feasible solution $(v,\mu,y,z)$,

(a) f is quasiconvex at v and $y^t g + z^t h$ is strictly pseudovonvex at v;

or
(b) $f$ is quasiconvex at $v$ and $y^t g + z^t h$ is strictly pseudoconvex at $v$;
or
(c) $f$ is quasiconvex at $v$, $y^t g$ is strictly pseudoconvex at $v$ and $z^t h$ is quasiconvex at $v$; or
(d) $\mu^t f$ is quasiconvex at $v$, $y^t g$ is strictly pseudoconvex at $v$ and $z^t h$ is quasiconvex at $v$,
then $f(x) \leq f(v)$.

**Proof.** (a) Suppose that there exist a (PE)-feasible solution $x$ and a (DE)-feasible solution $(v,\mu,y,z)$ such that $f(x) \leq f(v)$. Then, by the quasiconvexity of $f$ at $v$, $(x-v)^t \nabla f(v) \leq 0$. Since $\mu \geq 0$, we have

(26) $\quad (x - v)^t \nabla \mu^t f(v) \leq 0.$

Since $y^t g(x) \leq 0$ and $z^t h(x) = 0$, from (5) and (6), we have

$$y^t g(x) + z^t h(x) \leq y^t g(v) + z^t h(v)$$

Thus, by the strict pseudoconvexity of at $v$, we have

(27) $\quad (x - v)^t [\nabla y^t g(v) + \nabla z^t h(v)] < 0.$

From (26) and (27), we have

$$(x - v)^t [\nabla \mu^t g(v) + \nabla y^t g(v) + \nabla z^t h(v)] < 0,$$

which contradict (4). Hence the result follows.

(b) Suppose that there exist a (PE)- feasible solution $x$ and a (DE)-feasible $(v,u,y,z)$ such that $f(x) < f(v)$. Since $u > 0$,we have $\mu^t f(x) \leq \mu^t f(v)$ By the quasiconvexity of $\mu^t$ at $v$, we have

$$( x - v )^t \nabla \mu^t f(v) \leq 0.$$

By the same method as the proof of the part (a), we can prove the part(b).

(c) Suppose that there exist a (PE)-feasible solution $x$ and a (DE)-feasible solution $(v,u,y,z)$ such that $f(x) < f(v)$. then,by the quasiconvexity of $f$ at $v$,$(x - v)^t \nabla f(v) \leq 0$. Since $u > 0$, we have

(28) $( x - v )^t \nabla \mu^t f(v) \leq 0$

Since $z^t h(x) = 0$, from (6), we have $z^t h(x) \leq z^t h(v)$ and hence by the quasiconvexity of $z^t h$ at $v$, we have

(29) $(x - v)^t \nabla z^t h(v) \leq 0$

since $y^t g(x) \leq 0$, from(5), we have $y^t g(x) \leq y^t g(v)$. thus, by the strict pseudoconvexity of $y^t g$ at v, we have

(30) $(x - v)^t \nabla y^t g(v) \leq 0$

From (28),(29) and (30), we have

$$(x - v)^t[\nabla \mu^t f(v) + \nabla y^t g(v) + \nabla z^t h(v)] < 0,$$

which contradicts (4). Hence the result follows.

(d) Suppose that there esist a (PE)-feasible solution x and a (DE)-feasible solution (v,u,y,z) such that $f(x) \leq f(v)$. Since $u \geq 0$, we have $\mu^t f(x) \leq \mu^t f(v)$. Thus by the quasiconvexity of at v, we have

$$(x - v)^t \nabla \mu^t f(v) \leq 0.$$

By the same method as the proof of the part (c), we can prove the part (d).

**THEOREM 3.4.** Let x be an efficient solution for (PE) and the constraint function h be continuously differentiable at x. Then there exist $\mu \in R^p, y \in R^m$ and $z \in R^k$ such that $(x, \mu, y, z)$ is a (DE)-feasible solution and the objective values of (PE) and (DE) are equal. Moreover, if, for any (PE)-feasible solution x and any (DE)-feasible solution $(x, \mu, y, z)$,

(a) f is quasiconvex at v and $y^t g + z^t h$ is strictly pseudoconvex at v; or
(b) $\mu^t f$ is quasiconvex at v and $y^t g + z^t h$ is strictly pseudoconvex at v; or
(c) f is quasiconvex at v, $y^t g$ is strictly pseudoconvex at v and $z^t h$ is quasiconvex at v; or
(d) $\mu^t f$ is quasiconvex at v, $y^t g$ is strictly pseudoconvex at v and $z^t h$ is quasiconvex at v,
then $(x, \mu, y, z)$ is an efficient solution for (DE).

**Proof.** Since x is an efficient for (PE). x is a weakly efficient solution for (PE). By Theorem 2.2, there exist $\mu \in R^p, y \in R^m$ and $z \in R^k$ such that $\nabla \mu^t f(x) + \nabla y^t g(x) + \nabla z^t h(x) = 0, (\mu, y) \geq 0$ and $(\mu, y, z) \neq 0$ since $\nabla z^t h(x) = 0$, $(x, \mu, y, z)$ is a (DE)-feasible solution and clearly the objective values of (PE) and (DE) are equal.

By Theorem 3.3, $f(x) \leq f(v)$ for any (DE)-feasible solution $(v, \mu, y, z)$.

Since (x,$\mu$,y,z) is a (DE)-feasible solution, (x,$\mu$,y,z) is an efficient solution for (DE). Hence the result holds.

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