# OPTIMAL ERROR ANALYSIS OF THE $P$–VERSION UNDER QUADRATURE RULES

IK-SUNG KIM

## 1. Introduction

Let $\Omega$ be a closed and bounded polygonal domain in $R^2$, or a closed line segment in $R^1$ with boundary $\Gamma$, such that there exists an invertible mapping $T : \widehat{\Omega} \to \Omega$ with the following correspondence:

$$\widehat{x} \in \widehat{\Omega} \longleftrightarrow x = T(\widehat{x}) \in \Omega, \tag{1.1}$$

and

$$\widehat{t} \in U_p(\widehat{\Omega}) \longleftrightarrow t = \widehat{t} \circ T^{-1} \in U_p(\Omega), \tag{1.2}$$

where $\widehat{\Omega}$ denotes the corresponding reference elements $\widehat{I} = [-1, 1]$ and $\widehat{I} \times \widehat{I}$ in $R^1$ and $R^2$ respectively,

$$\begin{aligned} U_p(\widehat{\Omega}) \\ = \{ \widehat{t} : \widehat{t} \text{ is a polynomial of degree} \leq p \text{ in each variable on } \widehat{\Omega} \}, \end{aligned} \tag{1.3}$$

and

$$U_p(\Omega) = \{ t : \widehat{t} = t \circ T \in U_p(\widehat{\Omega}) \}. \tag{1.4}$$

We introduce Sobolev spaces

$$H^{m,p}(\Omega) \equiv \text{The completion of } \{u \in C^m(\Omega) : \|u\|_{m,p,\Omega} < \infty\}, \tag{1.5}$$

equipped with norm

$$\|u\|_{m,p,\Omega} = \left( \sum_{0 \leq |i| \leq m} \|\partial^i u\|^p_{0,p,\Omega} \right)^{1/p} \quad \text{if} \quad 1 \leq p < \infty, \tag{1.6}$$

$$\|u\|_{m,\infty,\Omega} = \max_{0 \leq |i| \leq m} \|\partial^i u\|_{0,\infty,\Omega}, \tag{1.7}$$

where $\| \cdot \|_{0,p,\Omega}$ is the usual $L_p(\Omega)$-norm, and the subscript $p$ may be dropped when $p = 2$.

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Now we define a space $H_0^m(\Omega) = \{u \in H^m(\Omega) : u \text{ vanishes on } \Gamma\}$, and consider the following model problem of non-constant coefficient elliptic equations:

Our model problem is to find $u \in H_0^1(\Omega)$, such that

$$(1.8) \qquad -\operatorname{div}(a\nabla u) = f \quad \text{in} \quad \Omega \subset R^2,$$

$$(1.9) \qquad -\frac{d}{dx}(a\frac{du}{dx}) = f \quad \text{in} \quad \Omega \subset R^1.$$

Here, for sake of simplicity to ensure a solution exists we assume

$$(1.10) \qquad 0 < A_1 \le a(x) \le A_2 \quad \text{for all} \quad x \in \Omega,$$

and

$$(1.11) \qquad f \in L_2(\Omega).$$

In addition, we also assume that there exists a constant $A$ such that

$$(1.12) \qquad \|T\|_{j,\infty,\widehat{\Omega}} \quad , \quad \|T^{-1}\|_{j,\infty,\Omega} \le A \quad \text{for} \quad 0 \le j \le M,$$

$$(1.13) \qquad \|\widehat{J}\|_{j,\infty,\widehat{\Omega}} \quad , \quad \|\widehat{J}^{-1}\|_{j,\infty,\Omega} \le A \quad \text{for} \quad 0 \le j \le M-1,$$

where $\widehat{J}$ and $\widehat{J}^{-1}$ denote the Jacobians of $T$ and $T^{-1}$ respectively, and $M \ge 1$. We note that $M$ must be large enough to ensure that the domain $\Omega$ is not too distorted, i.e., $T$ is smooth. For non-smooth mappings, (1.12) and (1.13) can still hold, but the constant $A$ may be very large.

By (1.12) and (1.13), as seen in theorem 4.3.2 of [6], we obtain the following correspondence:

For any $\alpha \in [1,\infty]$, $0 \le \beta \le M$,

$$(1.14) \qquad \widehat{t} \in W^{\beta,\alpha}(\widehat{\Omega}) \longleftrightarrow t = \widehat{t} \circ T^{-1} \in W^{\beta,\alpha}(\Omega)$$

with norm equivalence

$$(1.15) \qquad C_1\|t\|_{\beta,\alpha,\Omega} \le \|\widehat{t}\|_{\beta,\alpha,\widehat{\Omega}} \le C_2\|t\|_{\beta,\alpha,\Omega}.$$

Our problem (1.8)-(1.9) may be approximated by several numerical methods. In this paper we are interested in the $p$-version of the finite element method. The classical form of the finite element method, called the $h$-version, uses piecewise polynomials of a fixed degree p and decreases the mesh-size h to achieve accuracy. In the $p$-version, a fixed mesh is used while the degree p is increased for greater accuracy. The $h-p$ version is a combination of both. The standard $h$-version has been thoroughly investigated. But the $p$- and $h-p$ versions are recent developments. A survey of the $p$ -version's computational and theoretical characteristics may be found in [3]. Here, when we use the $p$-version of the finite element method without subdividing $\Omega$ the discrete variational form of (1.8)-(1.9) is to find $u_p \in S_{p,0}(\Omega)$ satisfying

$$B(u_p, v_p) = (f, v_p)_\Omega \quad \text{for all} \quad v_p \in S_{p,0}(\Omega), \tag{1.16}$$

where

$$B(u,v) = \int_\Omega a\nabla u \cdot \nabla v \, dx, \tag{1.17}$$

$$(f,v)_\Omega = \int_\Omega fv \, dx, \tag{1.18}$$

and

$$S_{p,0}(\Omega) = U_p(\Omega) \cap H_0^1(\Omega). \tag{1.19}$$

In [2] and [8], M. Suri obtained optimal error-estimates

$$\|u - u_p\|_{0,\Omega} \le C\, p^{-1} \|u - u_p\|_{1,\Omega} \tag{1.20}$$

and

$$\|u - u_p\|_{1,\Omega} \le C\, p^{-(r-1)} \|u\|_{r,\Omega} \quad \text{for all} \quad u \in H_0^r(\Omega),\ r \ge 1. \tag{1.21}$$

But, the above results follow under the assumption that $T$ is a sufficiently smooth mapping and all integrations in (1.16) are performed exactly. In practice, the integrals in (1.16) are seldom computed exactly. To compute the integrals in the variational form (1.16) of the discrete problem we need the numerical quadrature rule scheme. In this paper, when some numerical quadrature rules are used for calculating the integrations in the stiffness matrix and the load vector of (1.16) we give its variational form and derive the estimates of $u - \widetilde{u}_p$ in the $L_2(\Omega)$- and $H^1(\Omega)$-norm, where $\widetilde{u}_p$ is an approximation satisfying (2.5). In [7], the spectral element method has been introduced and Y. Maday point out the cases where overintegrations would be required. We also analyze the cases in which the overintegration may improve the accuracy of the approximation to allow for optimal results. In particular, we observe more general mapping $T : \widehat{\Omega} \to \Omega$ without subdividing the domain $\Omega$. This may have influence on the smoothness of the integrands in the variational form. Using Gauss-Legendre(G-L) quadrature rules some numerical experiments confirm the results.

## 2. Preliminaries

We consider numerical quadrature rules $I_k$ defined on the reference element $\widehat{\Omega}$ by

$$(2.1) \qquad I_k(\widehat{f}) = \sum_{i=1}^{n(k)} \widehat{w}_i^k \, \widehat{f}(\widehat{x}_i^k) \sim \int_{\widehat{\Omega}} \widehat{f}(\widehat{x})\, d\widehat{x},$$

where $k$ is a positive integer. Let $G_p = \{I_k\}$ be a family of quadrature rules $I_k$ with respect to $U_p(\widehat{\Omega})$, $p = 1, 2, 3, \cdots$, satisfying the following properties: For each $I_k \in G_p$,

(K1) $\widehat{w}_i^k > 0$ and $\widehat{x}_i^k \in \widehat{\Omega}$ for $i = 1, \cdots, n(k)$.

(K2) $I_k(\widehat{f}^{\,2}) \le C_1 \|\widehat{f}\|_{0,\widehat{\Omega}}^2$ for all $\widehat{f} \in U_p(\widehat{\Omega})$.

(K3) $C_2 \|\widetilde{f}\|_{0,\widehat{\Omega}}^2 \le I_k(\widetilde{f}^{\,2})$ for all $\widetilde{f} \in \widetilde{U}_p(\widehat{\Omega})$,

where $\widetilde{U}_p(\widehat{\Omega}) = \{ \dfrac{\partial \widehat{f}}{\partial \widehat{x}_i} : \widehat{f} \in U_p(\widehat{\Omega}) \} \subset U_p(\widehat{\Omega})$.

(K4) $I_k(\widehat{f}) = \int_{\widehat{\Omega}} \widehat{f}(\widehat{x})\, d\widehat{x}$ for all $\widehat{f} \in U_{d(k)}(\widehat{\Omega})$,

where $d(k) \ge \widetilde{d}(p) > 0$.

We also get a family $G_{p,\Omega} = \{I_{k,\Omega}\}$ of numerical quadrature rules with respect to $U_p(\Omega)$, which are defined on $\Omega$ by

$$(2.2) \qquad I_{k,\Omega}(f) = \sum_{i=1}^{n(k)} w_i^k f(x_i^k) = \sum_{i=1}^{n(k)} \widehat{w}_i^k \widehat{J}(\widehat{x}_i^k)(f \circ T)(\widehat{x}_i^k) = I_k(\widehat{J}\,\widehat{f}).$$

Now, we denote by DF the $n \times n$ Jacobian matrix of $F : R^n \to R^n$, and define two discrete inner products

$$(2.3) \qquad (u, v)_{l,\Omega} = I_{l,\Omega}(uv) \quad \text{on} \quad \Omega,$$

$$(2.4) \qquad (\widehat{u}, \widehat{v})_{l,\widehat{\Omega}} = I_l(\widehat{u}\,\widehat{v}) \quad \text{on} \quad \widehat{\Omega}.$$

Then, using quadrature rules $I_m$ and $I_l$ in $G_p$ we obtain the following actual problem of (1.16): To find $\widetilde{u}_p \in S_{p,0}(\Omega)$, such that

(2.5) $$B_{m,\Omega}(\widetilde{u}_p, v_p) = (f, v_p)_{l,\Omega} \quad \text{for all} \quad v_p \in S_{p,0}(\Omega),$$

where

(2.6) $$B_{m,\Omega}(\widetilde{u}_p, v_p) = \sum_{i,j=1}^{n} \left( \widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{\widetilde{u}}_p}{\partial \widehat{x}_i}, \frac{\partial \widehat{v}_p}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}},$$

(2.7) $$(f, v_p)_{l,\Omega} = (\widehat{J}\,\widehat{f}, \widehat{v}_p)_{l,\widehat{\Omega}},$$

and $\widehat{a}_{ij}$ denote the entries of the matrix $\widehat{J}(\widehat{DT^{-1}})(\widehat{DT^{-1}})^t$.

The following Lemmas will be used later

LEMMA 2.1. *For each integer $l \geq 0$, there exists a sequence of projections*

$$\Pi_p^l : H^l(\widehat{\Omega}) \to U_p(\widehat{\Omega}), \quad p = 1, 2, 3, \cdots, \quad \textit{such that}$$

(2.8) $$\Pi_p^l \widehat{v}_p = \widehat{v}_p \quad \textit{for all} \quad \widehat{v}_p \in U_p(\widehat{\Omega}),$$

(2.9) $$\|\widehat{u} - \Pi_p^l \widehat{u}\|_{s,\widehat{\Omega}} \leq C\, p^{-(r-s)} \|\widehat{u}\|_{r,\widehat{\Omega}} \quad \textit{for all} \quad \widehat{u} \in H^r(\widehat{\Omega}) \quad \textit{with} \quad 0 \leq s \leq l \leq r.$$

*Proof.* See [8, *Lemma* 3.1].

LEMMA 2.2. *There exists a sequence of projections*

$$P_p^1 : H_0^1(\Omega) \to S_{p,0}(\Omega), \quad p = 1, 2, 3, \cdots, \quad \textit{such that}$$

(2.10) $$\|u - P_p^1 u\|_{s,\Omega} \leq C\, p^{-(r-s)} \|u\|_{r,\Omega} \quad \textit{for all} \quad u \in H_0^r(\Omega) \quad \textit{with} \quad 0 \leq s \leq 1 < r.$$

*Proof.* See [8, *Theorem* 4.2].

LEMMA 2.3. *For $\widehat{\Omega} \subset R^n$, let $\widehat{u} \in H^s(\widehat{\Omega})$ with $s \geq n$. Then the projection $\Pi_p^n$ from Lemma 2.1 satisfies*

(2.11) $$\|\widehat{u} - \Pi_p^n \widehat{u}\|_{0,\infty,\widehat{\Omega}} \leq C\, p^{-(s-\frac{n}{2})} \|\widehat{u}\|_{s,\widehat{\Omega}}.$$

*Proof.* By interpolation results ( see [5, Theorem 3.2 ] and [ 4, Theorem 6.2.4 ] ) we have that

$$(2.12)\qquad \|\widehat{u}-\Pi_p^n\widehat{u}\|_{0,\infty,\widehat{\Omega}} \le C\,\|\widehat{u}-\Pi_p^n\widehat{u}\|^{\frac{1}{2}}_{\frac{n}{2}+\varepsilon,\widehat{\Omega}}\|\widehat{u}-\Pi_p^n\widehat{u}\|^{\frac{1}{2}}_{\frac{n}{2}-\varepsilon,\widehat{\Omega}}$$
$$\text{for}\quad 0<\varepsilon\le\tfrac{1}{2}.$$

We also have from Lemma 2.1 that

$$(2.13)\qquad \|\widehat{u}-\Pi_p^n\widehat{u}\|_{r,\widehat{\Omega}} \le C\,p^{-(s-r)}\|\widehat{u}\|_{s,\widehat{\Omega}} \quad \text{for}\quad 0\le r\le n\le s.$$

Hence, taking with $r=\frac{n}{2}+\varepsilon$ and $r=\frac{n}{2}-\varepsilon$ in (2.13) we obtain

$$\|\widehat{u}-\Pi_p^n\widehat{u}\|^{\frac{1}{2}}_{\frac{n}{2}+\varepsilon,\widehat{\Omega}}\|\widehat{u}-\Pi_p^n\widehat{u}\|^{\frac{1}{2}}_{\frac{n}{2}-\varepsilon,\widehat{\Omega}} \le C\,p^{-(s-\frac{n}{2})}\|\widehat{u}\|_{s,\widehat{\Omega}},$$

which completes the proof from (2.12).

## 3. Error estimates under numerical quadrature rules and mappings

First we shall estimate $\|u-\widetilde{u}_p\|_{1,\Omega}$ which depends on several separate terms. The first dependence is on the error $\|u-u_p\|_{1,\Omega}$ with respect to the mapping $T$. Next, the error will depend upon the smoothness of $\widehat{a}$, $\widehat{a}_{ij}$ and $\widehat{f}$ with the Jacobian $\widehat{J}$ of $T$.

LEMMA 3.1. *Let $u$ be the exact solution of (1.8)-(1.9) and $\widetilde{u}_p$ an approximation of $u$ which satisfies (2.5). Then there exists a constant $C$ independent of $m$, $l$ such that*

$$(3.1)\qquad \|u-\widetilde{u}_p\|_{1,\Omega} \le C\Big[\inf_{u_p\in S_{p,0}(\Omega)}\Big\{\|u-u_p\|_{1,\Omega} + \sup_{w_p\in S_{p,0}(\Omega)}\frac{|B(u_p,w_p)-B_{m,\Omega}(u_p,w_p)|}{\|w_p\|_{1,\Omega}}\Big\} + \sup_{w_p\in S_{p,0}(\Omega)}\frac{|(f,w_p)_\Omega-(f,w_p)_{l,\Omega}|}{\|w_p\|_{1,\Omega}}\Big].$$

*Proof.* It is similar to the technique in [6, Theorem 4.1.1 ].

In Lemma 3.1, the third factor that $\|u-\widetilde{u}_p\|_{1,\Omega}$ depends upon is the smoothness of $\widehat{f}$ and $\widehat{J}$ with the mapping $T$. In this connection, we shall use the following Lemma.

LEMMA 3.2. *Let $I_l \in G_p$ be a quadrature rule on $\widehat{\Omega} \subset R^n$ which satisfies $d(l) - p - 1 > 0$, and let $\widehat{f} \in H^{\gamma}(\widehat{\Omega})$ and $\widehat{J} \in H^{\delta}(\widehat{\Omega})$ with $\min(\gamma, \delta) \geq n$. Then, for any $w_p \in S_{p,0}(\Omega)$ we have the following estimate*

$$
\begin{aligned}
(3.2) \qquad & \frac{|(f, w_p)_\Omega - (f, w_p)_{l,\Omega}|}{\|w_p\|_{1,\Omega}} \\
& \leq C\{\, q^{-(\gamma - \frac{n}{2})} \|\widehat{f}\|_{\gamma,\widehat{\Omega}} (\, \|\widehat{J}\|_{0,\infty,\widehat{\Omega}} + \|\widehat{J}\|_{\delta,\widehat{\Omega}}\,) \\
& \quad + (d(l) - p - q)^{-(\delta - \frac{n}{2})} \|\widehat{J}\|_{\delta,\widehat{\Omega}} \,(\, \|\widehat{f}\|_{0,\infty,\widehat{\Omega}} + \|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,)\,\},
\end{aligned}
$$

*where $q$ is a positive integer with $d(l) - p - q > 0$ and $C$ is independent of $l, p$ and $q$.*

*Proof.* Since $d(l) - p - 1 > 0$ there exists a positive integer $q$ such that $d(l) - p - q > 0$. For arbitrary $\widehat{w}_1 \in U_{d(l)-p-q}(\widehat{\Omega})$ and $\widehat{w}_2 \in U_q(\widehat{\Omega})$ we let $\widehat{w} = \widehat{w}_1 \widehat{w}_2 \in U_{d(l)-p}(\widehat{\Omega})$. Then, due to (K4) it follows that

$$
(3.3) \qquad (\widehat{w}, \widehat{w}_p)_{l,\widehat{\Omega}} - (\widehat{w}, \widehat{w}_p)_{\widehat{\Omega}} = 0.
$$

Since $(f, w_p)_\Omega = (\widehat{J}\widehat{f}, \widehat{w}_p)_{\widehat{\Omega}}$ and $(f, w_p)_{l,\Omega} = (\widehat{J}\widehat{f}, \widehat{w}_p)_{l,\widehat{\Omega}}$

$$
\begin{aligned}
(3.4) \qquad & |(f, w_p)_\Omega - (f, w_p)_{l,\Omega}| \\
& \leq |(\widehat{J}\widehat{f}, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{w}, \widehat{w}_p)_{\widehat{\Omega}}| + |(\widehat{w}, \widehat{w}_p)_{l,\widehat{\Omega}} - (\widehat{J}\widehat{f}, \widehat{w}_p)_{l,\widehat{\Omega}}|.
\end{aligned}
$$

By the Schwarz inequality we obtain

$$
\begin{aligned}
(3.5) \qquad & |(\widehat{J}\widehat{f}, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{w}, \widehat{w}_p)_{\widehat{\Omega}}| \\
& \leq |(\widehat{J}\widehat{f} - \widehat{J}\widehat{w}_2, \widehat{w}_p)_{\widehat{\Omega}}| + |(\widehat{J}\widehat{w}_2 - \widehat{w}_1\widehat{w}_2, \widehat{w}_p)_{\widehat{\Omega}}| \\
& \leq \|\widehat{J}(\widehat{f} - \widehat{w}_2)\|_{0,\widehat{\Omega}} \|\widehat{w}_p\|_{0,\widehat{\Omega}} + \|(\widehat{J} - \widehat{w}_1)\widehat{w}_2\|_{0,\widehat{\Omega}} \|\widehat{w}_p\|_{0,\widehat{\Omega}} \\
& \leq (\, \|\widehat{J}\|_{0,\widehat{\Omega}} \|\widehat{f} - \widehat{w}_2\|_{0,\infty,\widehat{\Omega}} + \|\widehat{J} - \widehat{w}_1\|_{0,\infty,\widehat{\Omega}} \|\widehat{w}_2\|_{0,\widehat{\Omega}}\,) \|\widehat{w}_p\|_{0,\widehat{\Omega}} \;.
\end{aligned}
$$

Taking $\widehat{w}_1 = \Pi^n_{d(l)-p-q}(\widehat{J})$ and $\widehat{w}_2 = \Pi^n_q(\widehat{f})$ in Lemma 2.3 we have

$$
(3.6) \qquad \|\widehat{f} - \widehat{w}_2\|_{0,\infty,\widehat{\Omega}} \leq C\, q^{-(\gamma - \frac{n}{2})} \|\widehat{f}\|_{\gamma,\widehat{\Omega}} \,,
$$

and

$$(3.7) \qquad \|\widehat{J} - \widehat{w}_1\|_{0,\infty,\widehat{\Omega}} \le C\,(d(l)-p-q)^{-(\delta-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,.$$

Moreover, by the triangle inequality and from Lemma 2.1

$$(3.8) \qquad \begin{aligned} \|\widehat{w}_2\|_{0,\widehat{\Omega}} &\le \|\widehat{f}\|_{0,\widehat{\Omega}} + \|\widehat{f} - \widehat{w}_2\|_{0,\widehat{\Omega}} \\ &\le C\,\{\,\|\widehat{f}\|_{\gamma,\widehat{\Omega}} + q^{-\gamma}\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,\} \\ &\le C\,\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,, \end{aligned}$$

and obviously

$$(3.9) \qquad \|\widehat{J}\|_{0,\widehat{\Omega}} \le C\,\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,.$$

Hence, by substituting the above results in (3.5) we have

$$(3.10) \qquad \begin{aligned} &|\,(\widehat{J}\,\widehat{f},\widehat{w}_p)_{\widehat{\Omega}} - (\widehat{w},\widehat{w}_p)_{\widehat{\Omega}}\,| \\ &\le C\,\{q^{-(\gamma-\frac{n}{2})} + (d(l)-p-q)^{-(\delta-\frac{n}{2})}\}\,\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}\,. \end{aligned}$$

Similarly, we can estimate the last term of the right side in (3.4), which can be rewritten as

$$(3.11) \qquad \begin{aligned} &|\,(\widehat{J}\,\widehat{f},\widehat{w}_p)_{l,\widehat{\Omega}} - (\widehat{w},\widehat{w}_p)_{l,\widehat{\Omega}}\,| \\ &\le |\,(\widehat{J}\,\widehat{f},\widehat{w}_p)_{l,\widehat{\Omega}} - (\widehat{J}\,\widehat{w}_2,\widehat{w}_p)_{l,\widehat{\Omega}}\,| + |\,(\widehat{J}\,\widehat{w}_2,\widehat{w}_p)_{l,\widehat{\Omega}} - (\widehat{w}_1\widehat{w}_2,\widehat{w}_p)_{l,\widehat{\Omega}}\,| \\ &= |\,(\widehat{J}(\widehat{f}-\widehat{w}_2),\widehat{w}_p)_{l,\widehat{\Omega}}\,| + |\,(\widehat{w}_2(\widehat{J}-\widehat{w}_1),\widehat{w}_p)_{l,\widehat{\Omega}}\,|\,. \end{aligned}$$

Using the Schwarz inequality, we have from (3.6) and (K2) that

$$(3.12) \quad \begin{aligned} |\,(\widehat{J}(\widehat{f}-\widehat{w}_2),\widehat{w}_p)_{l,\widehat{\Omega}}\,| &\le (\widehat{J}(\widehat{f}-\widehat{w}_2),\,\widehat{J}(\widehat{f}-\widehat{w}_2))^{\frac{1}{2}}_{l,\widehat{\Omega}}\,(\widehat{w}_p,\widehat{w}_p)^{\frac{1}{2}}_{l,\widehat{\Omega}} \\ &\le C\,\|\widehat{J}\|_{0,\infty,\widehat{\Omega}}\,\|\widehat{f}-\widehat{w}_2\|_{0,\infty,\widehat{\Omega}}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}} \\ &\le C\,q^{-(\gamma-\frac{n}{2})}\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,\|\widehat{J}\|_{0,\infty,\widehat{\Omega}}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}\,. \end{aligned}$$

Moreover, from (3.6) and (3.7) we also obtain

$$(3.13) \qquad \begin{aligned} &|\,(\widehat{w}_2(\widehat{J}-\widehat{w}_1),\widehat{w}_p)_{l,\widehat{\Omega}}\,| \\ &\le (\widehat{w}_2(\widehat{J}-\widehat{w}_1),\widehat{w}_2(\widehat{J}-\widehat{w}_1))^{\frac{1}{2}}_{l,\widehat{\Omega}}\,(\widehat{w}_p,\widehat{w}_p)^{\frac{1}{2}}_{l,\widehat{\Omega}} \\ &\le C\,\|\widehat{J}-\widehat{w}_1\|_{0,\infty,\widehat{\Omega}}\,\|\widehat{w}_2\|_{0,\infty,\widehat{\Omega}}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}} \end{aligned}$$

$$\leq C\,\|\widehat{J}-\widehat{w}_1\|_{0,\infty,\widehat{\Omega}}\,(\|\widehat{f}\|_{0,\infty,\widehat{\Omega}}+\|\widehat{f}-\widehat{w}_2\|_{0,\infty,\widehat{\Omega}})\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}$$
$$\leq C\,\{\,(d(l)-p-q)^{-(\delta-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,\|\widehat{f}\|_{0,\infty,\widehat{\Omega}}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}$$
$$+\,(d(l)-p-q)^{-(\delta-\frac{n}{2})}q^{-(\gamma-\frac{n}{2})}\,\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}\,\}.$$

Hence, combining (3.12) and (3.13) we estimate

$$(3.14)\quad |\,(\widehat{J}\,\widehat{f},\widehat{w}_p)_{l,\widehat{\Omega}}-(\widehat{w},\widehat{w}_p)_{l,\widehat{\Omega}}\,|$$
$$\leq C\,\{q^{-(\gamma-\frac{n}{2})}\|\widehat{J}\|_{0,\infty,\widehat{\Omega}}\,\|\widehat{f}\|_{\gamma,\widehat{\Omega}}$$
$$+\,(d(l)-p-q)^{-(\delta-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,\|\widehat{f}\|_{0,\infty,\widehat{\Omega}}$$
$$+\,q^{-(\gamma-\frac{n}{2})}(d(l)-p-q)^{-(\delta-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,\}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}\,.$$

Since the last term of the right side in (3.14) is dominated by the terms in (3.10) we derive

$$(3.15)\quad |\,(f,w_p)_{\Omega}-(f,w_p)_{l,\Omega}\,|$$
$$\leq C\{\,q^{-(\gamma-\frac{n}{2})}\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,(\,\|\widehat{J}\|_{0,\infty,\widehat{\Omega}}+\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,)$$
$$+\,(d(l)-p-q)^{-(\delta-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,(\,\|\widehat{f}\|_{0,\infty,\widehat{\Omega}}+\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,\}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}\,.$$

It is obvious from (1.15) that

$$(3.16)\quad \|\widehat{w}_p\|_{0,\widehat{\Omega}}\;\leq\;C\,\|\widehat{w}_p\|_{1,\widehat{\Omega}}\;\leq\;C\,\|w_p\|_{1,\Omega}\,.$$

The Lemma follows from dividing with $\|w_p\|_{1,\Omega}$ .

Now, we give the following Lemma which can be used for estimating the middle term in (3.1).

LEMMA 3.3. *Let* $\widehat{u}_p,\,\widehat{w}_p\in U_p(\widehat{\Omega})$ *and* $\widehat{f}\in L_\infty(\widehat{\Omega})$. *Then, for all* $\widehat{v}_q\in U_q(\widehat{\Omega})$, $\widehat{f}_r\in U_r(\widehat{\Omega})$ *with* $0<q\leq p$ *and* $r=d(m)-p-q>0$ *we have*

$$(3.17)\quad |\,(\widehat{f}\,\widehat{u}_p,\widehat{w}_p)_{\widehat{\Omega}}-(\widehat{f}\,\widehat{u}_p,\widehat{w}_p)_{m,\widehat{\Omega}}\,|$$
$$\leq\;C\,\{\,\|\widehat{f}_r\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p-\widehat{v}_q\|_{0,\widehat{\Omega}}+\|\widehat{f}-\widehat{f}_r\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p\|_{0,\widehat{\Omega}}\,\}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}\quad ,$$

*where* $C$ *is independent of* $p$, $q$ *and* $m$.

*Proof.* For any $\widehat{f}_r\in U_r(\widehat{\Omega})$ we have

$$(3.18)\qquad |(\hat f\,\hat u_p,\hat w_p)_{\hat\Omega}-(\hat f\,\hat u_p,\hat w_p)_{m,\hat\Omega}|$$
$$\le |(\hat f\,\hat u_p,\hat w_p)_{\hat\Omega}-(\hat f_r\hat u_p,\hat w_p)_{\hat\Omega}|+|(\hat f_r\hat u_p,\hat w_p)_{\hat\Omega}-(\hat f_r\hat u_p,\hat w_p)_{m,\hat\Omega}|$$
$$+|(\hat f_r\hat u_p,\hat w_p)_{m,\hat\Omega}-(\hat f\,\hat u_p,\hat w_p)_{m,\hat\Omega}|.$$

Thank to (K4),

$$(3.19)\qquad (\hat f_r\hat v_q,\hat w_p)_{\hat\Omega}-(\hat f_r\hat v_q,\hat w_p)_{m,\hat\Omega}=0\quad\text{for any}\quad \hat v_q\in U_q(\hat\Omega).$$

Hence,

$$(3.20)\qquad |(\hat f_r\hat u_p,\hat w_p)_{\hat\Omega}-(\hat f_r\hat u_p,\hat w_p)_{m,\hat\Omega}|$$
$$\le |(\hat f_r\hat u_p,\hat w_p)_{\hat\Omega}-(\hat f_r\hat v_q,\hat w_p)_{\hat\Omega}|+|(\hat f_r\hat v_q,\hat w_p)_{m,\hat\Omega}-(\hat f_r\hat u_p,\hat w_p)_{m,\hat\Omega}|.$$

By the Schwarz inequality we obtain

$$(3.21)\qquad |(\hat f_r\hat u_p,\hat w_p)_{\hat\Omega}-(\hat f_r\hat v_q,\hat w_p)_{\hat\Omega}|$$
$$\le (\hat f_r(\hat u_p-\hat v_q),\hat f_r(\hat u_p-\hat v_q))^{\frac12}_{\hat\Omega}(\hat w_p,\hat w_p)^{\frac12}_{\hat\Omega}$$
$$\le C\|\hat f_r\|_{0,\infty,\hat\Omega}\|\hat u_p-\hat v_q\|_{0,\hat\Omega}\|\hat w_p\|_{0,\hat\Omega}\ .$$

Also, from (K2) we have

$$(3.22)\qquad |(\hat f_r\hat v_q,\hat w_p)_{m,\hat\Omega}-(\hat f_r\hat u_p,\hat w_p)_{m,\hat\Omega}|$$
$$\le (\hat f_r(\hat u_p-\hat v_q),\hat f_r(\hat u_p-\hat v_q))^{\frac12}_{m,\hat\Omega}(\hat w_p,\hat w_p)^{\frac12}_{m,\hat\Omega}$$
$$\le C\|\hat f_r\|_{0,\infty,\hat\Omega}(\hat u_p-\hat v_q,\hat u_p-\hat v_q)^{\frac12}_{m,\hat\Omega}(\hat w_p,\hat w_p)^{\frac12}_{m,\hat\Omega}$$
$$\le C\|\hat f_r\|_{0,\infty,\hat\Omega}\|\hat u_p-\hat v_q\|_{0,\hat\Omega}\|\hat w_p\|_{0,\hat\Omega}\ .$$

Hence, combining (3.21) and (3.22) we estimate

$$(3.23)\qquad |(\hat f_r\hat u_p,\hat w_p)_{\hat\Omega}-(\hat f_r\hat u_p,\hat w_p)_{m,\hat\Omega}|$$
$$\le C\|\hat f_r\|_{0,\infty,\hat\Omega}\|\hat u_p-\hat v_q\|_{0,\hat\Omega}\|\hat w_p\|_{0,\hat\Omega}\ .$$

Similarly, since $\hat f\in L_\infty(\hat\Omega)$ we obtain

$$(3.24)\qquad |(\hat f\,\hat u_p,\hat w_p)_{\hat\Omega}-(\hat f_r\hat u_p,\hat w_p)_{\hat\Omega}|$$

$$\leq ((\hat{f} - \hat{f}_r)\hat{u}_p, (\hat{f} - \hat{f}_r)\hat{u}_p)^{\frac{1}{2}}_{\hat{\Omega}} (\hat{w}_p, \hat{w}_p)^{\frac{1}{2}}_{\hat{\Omega}}$$
$$\leq C \|\hat{f} - \hat{f}_r\|_{0,\infty,\hat{\Omega}} \|\hat{u}_p\|_{0,\hat{\Omega}} \|\hat{w}_p\|_{0,\hat{\Omega}} ,$$

and

$$|(\hat{f}_r \hat{u}_p, \hat{w}_p)_{m,\hat{\Omega}} - (\hat{f}\,\hat{u}_p, \hat{w}_p)_{m,\hat{\Omega}}| \tag{3.25}$$
$$\leq ((\hat{f}_r - f)\hat{u}_p, (\hat{f}_r - \hat{f})\hat{u}_p)^{\frac{1}{2}}_{m,\hat{\Omega}} (\hat{w}_p, \hat{w}_p)^{\frac{1}{2}}_{m,\hat{\Omega}}$$
$$\leq C \|\hat{f}_r - \hat{f}\|_{0,\infty,\hat{\Omega}} (\hat{u}_p, \hat{u}_p)^{\frac{1}{2}}_{m,\hat{\Omega}} (\hat{w}_p, \hat{w}_p)^{\frac{1}{2}}_{m,\hat{\Omega}}$$
$$\leq C \|\hat{f}_r - \hat{f}\|_{0,\infty,\hat{\Omega}} \|\hat{u}_p\|_{0,\hat{\Omega}} \|\hat{w}_p\|_{0,\hat{\Omega}} .$$

The Lemma follows from (3.23), (3.24), (3.25) and (3.18).

For any $\hat{f} \in H^r(\hat{\Omega})$ with $\hat{\Omega} \subset R^n$ and $r \geq n$ we denote

$$K_s(\hat{f}) = \|\Pi^n_s \hat{f}\|_{0,\infty,\hat{\Omega}} . \tag{3.26}$$

Then, we easily see from Lemma 2.1 that

$$K_s(\hat{f}) \leq C \{ \|\hat{f}\|_{0,\infty,\hat{\Omega}} + s^{-(r-\frac{n}{2})} \|\hat{f}\|_{r,\hat{\Omega}} \} \tag{3.27}$$
$$\leq C \{ \|\hat{f}\|_{0,\infty,\hat{\Omega}} + \|\hat{f}\|_{r,\hat{\Omega}} \} .$$

Let us define

$$M_{p,q} = \max_{i,j} \|\hat{a}_{ij}\|_{p,q,\hat{\Omega}} , \tag{3.28}$$

where the subscript $q$ will be omitted when $q = 2$.

LEMMA 3.4. *Let $I_m \in G_p$ be a quadrature rule defined on $\hat{\Omega} \subset R^n$, which satisfies $d(m) - p - 1 > 0$. Let $\hat{u} \in H^{\sigma}(\hat{\Omega})$, $\hat{a} \in H^{\alpha}(\hat{\Omega})$, $\hat{J} \in H^{\delta}(\hat{\Omega})$ and $\hat{a}_{ij} \in H^{\rho}(\hat{\Omega})$ for $i,j = 1,\cdots,n$, such that $k = \min(\alpha,\rho) \geq n$. Then, for any $w_p \in S_{p,0}(\Omega)$ and an approximation $u_p$ which satisfies (1.16) we have*

$$\frac{|B(u_p, w_p) - B_{m,\Omega}(u_p, w_p)|}{\|w_p\|_{1,\Omega}} \tag{3.29}$$

$$\leq C\{q^{-(\sigma-1)}\|\widehat{u}\|_{\sigma,\widehat{\Omega}} + r^{-(k-\frac{n}{2})}\|\widehat{a}\|_{\alpha,\widehat{\Omega}} M_\rho \|\widehat{u}\|_{1,\widehat{\Omega}}\},$$

*where* $q$ *is a positive integer such that* $0 < q \leq p$ *and* $r = d(m) - p - q > 0$.

*Proof.* For arbitrary $w_p \in S_{p,0}(\Omega)$ we have

$$\begin{aligned} &| B(u_p, w_p) - B_{m,\Omega}(u_p, w_p)| \\ &\leq C \max_{i,j} \left| \left(\widehat{a}\,\widehat{a}_{ij}\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p}{\partial \widehat{x}_j}\right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a}_{ij}\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p}{\partial \widehat{x}_j}\right)_{m,\widehat{\Omega}} \right|. \end{aligned} \tag{3.30}$$

For any $\widehat{a}_{ij}$ $i, j = 1, \cdots, n$ we let $q$ be any integer such that $0 < q \leq p$ and $r = d(m) - p - q > 0$. Then, since $\widehat{a}\,\widehat{a}_{ij} \in L_\infty(\widehat{\Omega})$, due to Lemma 3.3 with $\widehat{v}_q = \frac{\partial}{\partial \widehat{x}_i}(\Pi_q^1 \widehat{u}_p)$ and $\widehat{f}_r = \Pi_r^n(\widehat{a}\,\widehat{a}_{ij})$, we have

$$\begin{aligned} &\left| \left(\widehat{a}\,\widehat{a}_{ij}\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p}{\partial \widehat{x}_j}\right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a}_{ij}\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p}{\partial \widehat{x}_j}\right)_{m,\widehat{\Omega}} \right| \\ &\leq C\{ \|\Pi_r^n(\widehat{a}\,\widehat{a}_{ij})\|_{0,\infty,\widehat{\Omega}} \left\|\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i} - \frac{\partial}{\partial \widehat{x}_i}(\Pi_q^1 \widehat{u}_p)\right\|_{0,\widehat{\Omega}} \\ &\quad + \|\widehat{a}\,\widehat{a}_{ij} - \Pi_r^n(\widehat{a}\,\widehat{a}_{ij})\|_{0,\infty,\widehat{\Omega}} \left\|\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}\right\|_{0,\widehat{\Omega}}\} \left\|\frac{\partial \widehat{w}_p}{\partial \widehat{x}_j}\right\|_{0,\widehat{\Omega}}. \end{aligned} \tag{3.31}$$

Using Lemma 2.1 we easily see from the boundedness of $\Pi_q^1$ that

$$\begin{aligned} &\left\|\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i} - \frac{\partial}{\partial \widehat{x}_i}(\Pi_q^1 \widehat{u}_p)\right\|_{0,\widehat{\Omega}} \\ &\leq C\|\widehat{u}_p - \Pi_q^1 \widehat{u}_p\|_{1,\widehat{\Omega}} \leq C q^{-(\sigma-1)}\|\widehat{u}\|_{\sigma,\widehat{\Omega}}. \end{aligned} \tag{3.32}$$

Also, clearly

$$\left\|\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}\right\|_{0,\widehat{\Omega}} \leq C\|\widehat{u}_p\|_{1,\widehat{\Omega}} \leq C\|\widehat{u}\|_{1,\widehat{\Omega}}, \tag{3.33}$$

and

$$\left\|\frac{\partial \widehat{w}_p}{\partial \widehat{x}_j}\right\|_{0,\widehat{\Omega}} \leq C\|\widehat{w}_p\|_{1,\widehat{\Omega}}. \tag{3.34}$$

Moreover, since $\widehat{a}\,\widehat{a}_{ij} \in H^k(\widehat{\Omega})$ with $k = \min(\alpha, \rho) \geq n$ we obtain from Lemma 2.3 that

$$\text{(3.35)} \qquad \|\widehat{a}\,\widehat{a}_{ij} - \Pi_r^n(\widehat{a}\,\widehat{a}_{ij})\|_{0,\infty,\widehat{\Omega}} \le C\, r^{-(k-\frac{n}{2})}\|\widehat{a}\|_{\alpha,\widehat{\Omega}} M_\rho \ .$$

So, from (3.32)-(3.35) and since $\|\Pi_r^n(\widehat{a}\,\widehat{a}_{ij})\|_{0,\infty,\widehat{\Omega}}$ is bounded, we have

$$\text{(3.36)} \qquad \max_{i,j} \left| \left( \widehat{a}\,\widehat{a}_{ij}\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p}{\partial \widehat{x}_j} \right)_{\widehat{\Omega}} - \left( \widehat{a}\,\widehat{a}_{ij}\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} \right|$$
$$\le C\left\{ q^{-(\sigma-1)}\|\widehat{u}\|_{\sigma,\widehat{\Omega}} + r^{-(k-\frac{n}{2})}\|\widehat{a}\|_{\alpha,\widehat{\Omega}} M_\rho \|\widehat{u}\|_{1,\widehat{\Omega}} \right\} \|\widehat{w}_p\|_{1,\widehat{\Omega}} \ .$$

Since $\|\widehat{w}_p\|_{0,\widehat{\Omega}} \le C\|\widehat{w}_p\|_{1,\widehat{\Omega}} \le C\,\|w_p\|_{1,\Omega}$, the Lemma follows from dividing by $\|w_p\|_{1,\Omega}$.

By a direct application of (1.21) and Lemma 3.2, 3.4 to Lemma 3.1 we obtain the following Theorem which gives an asymptotic $H^1(\Omega)$-norm estimate for the rate of convergence with using numerical quadrature rules and the mapping $T : \widehat{\Omega} \to \Omega \subset R^n$.

THEOREM 3.5. *For any numerical quadrature rules* $I_m, I_l \in G_p$ *and for any mapping* $T : \widehat{\Omega} \to \Omega \subset R^n$ *which satisfies (1.12)-(1.13), we assume that* $\widehat{u} \in H^\sigma(\widehat{\Omega})$, $\widehat{a} \in H^\alpha(\widehat{\Omega})$, $\widehat{J} \in H^\delta(\widehat{\Omega})$, $\widehat{f} \in H^\gamma(\widehat{\Omega})$ *and* $\widehat{a}_{ij} \in H^\rho(\widehat{\Omega})$ *for each* $i, j = 1, \cdots, n$ *with* $\min(\alpha, \gamma, \delta, \rho) \ge n$. *Then, for any positive integers* $q_1, q_2$ *such that* $0 < q_2 \le d(l) - p - 1$ *and* $0 < q_1 \le \min(d(m) - p - 1,\, p)$, *we have*

$$\text{(3.37)} \qquad \|u - \widetilde{u}_p\|_{1,\Omega} \le C\{ {q_1}^{-(\sigma-1)}\|\widehat{u}\|_{\sigma,\widehat{\Omega}}$$
$$+ {r_1}^{-(k-\frac{n}{2})}\|\widehat{a}\|_{\alpha,\widehat{\Omega}} M_\rho \|\widehat{u}\|_{1,\widehat{\Omega}}$$
$$+ {q_2}^{-(\gamma-\frac{n}{2})}\|\widehat{f}\|_{\gamma,\widehat{\Omega}} (\|\widehat{J}\|_{0,\infty,\widehat{\Omega}} + \|\widehat{J}\|_{\delta,\widehat{\Omega}})$$
$$+ {r_2}^{-(\delta-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}} (\|\widehat{f}\|_{0,\infty,\widehat{\Omega}} + \|\widehat{f}\|_{\gamma,\widehat{\Omega}}) \},$$

*where* $k = \min(\alpha, \rho)$, $r_2 = d(l) - p - q_2$ *and* $r_1 = d(m) - p - q_1$.

We see from Theorem 3.5 that the rate of convergence is essentially given by

$$\text{(3.38)} \qquad O(\ {q_1}^{-(\sigma-1)} + \ (d(m) - p - q_1)^{-(k-\frac{n}{2})}$$
$$+\ {q_2}^{-(\gamma-\frac{n}{2})} + (d(l) - p - q_2)^{-(\delta-\frac{n}{2})}\ ).$$

If $m$, $l$ and $q_2$ are large enough with $q_1 = p$, then the rate of convergence is asymptotically $O(p^{-(\sigma-1)})$, which coincides with that of (1.21). In the case where $\widehat{a}, \widehat{a}_{ij}, \widehat{f}$ and $\widehat{J}$ are sufficiently smooth, i.e., $k$ and $\gamma$ are large enough, even when $d(m) \approx 2p+1$ with $q_1 = p$ and $d(l) \approx p+2$ with $q_2 \approx 1$ the first term in (3.38) may dominate, so that the rate of convergence is asymptotically $O(p^{-(\sigma-1)})$ which is the same that of $\|u-u_p\|_{1,\Omega}$. More precisely, in G-L quadrature rules, using $I_m$ and $I_l$ with $(p+1)$-point and $p$-point G-L rules respectively we would obtain an asymptotic rate $O(p^{-(\sigma-1)})$.

When one of $\widehat{a}\,\widehat{a}_{ij}$ and $\widehat{J}\,\widehat{f}$ is not smooth enough, either because one of them is not smooth in the original problem or because a non-smooth mapping $T$ is used, the first term ${q_1}^{-(\sigma-1)}$ may be dominated by one of the other terms. In this situation, using an overintegration with a sufficient number of $m$ or $l$ we may reduce the error $\|u-\widetilde{u}_p\|_{1,\Omega}$ until the first term dominates again. In practice, when $\widehat{a}\,\widehat{a}_{ij}$ is not smooth we may increase the value of $d(m)$ with $q_1 \approx p$. When $\widehat{J}\,\widehat{f}$ is not sufficiently smooth we also increase both of $d(l)$ and $q_2$.

We now estimate the $L_2(\Omega)$- error. To estimate the error $\|u-\widetilde{u}_p\|_{0,\Omega}$ we start with the following Lemma.

LEMMA 3.6. *Let $u$ be the exact solution of (1.8)-(1.9) and $u_p$ the p-version solution of (1.16). Then, for an approximate solution $\widetilde{u}_p$ of $u_p$ which satisfies (2.5) we have*

(3.39)
$$\begin{aligned}\|u-\widetilde{u}_p\|_{0,\Omega} &\le \|u-u_p\|_{0,\Omega} \\ &+ \sup_{w_p\in S_{p,0}(\Omega)} \frac{1}{\|w_p\|_{0,\Omega}}(|B(\widetilde{u}_p,w)-B_{m,\Omega}(\widetilde{u}_p,w)| \\ &+ |(f,w)_\Omega-(f,w)_{l,\Omega}|),\end{aligned}$$

*where for each $w_p \in S_{p,0}(\Omega)$, $w \in S_{p,0}(\Omega)$ denotes the solution of discrete variational problem:*

(3.40) $$B(w,v_p) = (w_p,v_p)_\Omega \quad \textit{for all} \quad v_p \in S_{p,0}(\Omega).$$

*proof.* By the triangle inequality we have

(3.41) $$\|u-\widetilde{u}_p\|_{0,\Omega} \le \|u-u_p\|_{0,\Omega} + \|u_p-\widetilde{u}_p\|_{0,\Omega}.$$

where $q$ is a positive integer such that $0 < q \le p$ and $r = d(m)-p-q > 0$.

*proof.* For $w \in S_{p,0}(\Omega)$ we have

$$(3.49)\qquad \begin{aligned} &|\, B(\widetilde{u}_p, w) - B_{m,\Omega}(\widetilde{u}_p, w)\,| \\ &\le C\,\{ \max_{ij} | \left(\widehat{a}\widehat{a}_{ij}\frac{\partial\widehat{\widetilde{u}}_p}{\partial\widehat{x}_i}, \frac{\partial\widehat{w}}{\partial\widehat{x}_j}\right)_{\widehat{\Omega}} - \left(\widehat{a}\widehat{a}_{ij}\frac{\partial\widehat{\widetilde{u}}_p}{\partial\widehat{x}_i}, \frac{\partial\widehat{w}}{\partial\widehat{x}_j}\right)_{m,\widehat{\Omega}} |\}. \end{aligned}$$

Let $q$ be any integer such that $0 < q \le p$ and $r = d(m) - p - q > 0$. Then, for any $i = 1, \cdots, n$, due to Lemma 3.3 with $\widehat{f}_r = \Pi_r^n \widehat{a}\widehat{a}_{ij}$ and $\widehat{v}_q \in U_q(\widehat{\Omega})$, we have

$$(3.50)\qquad \begin{aligned} &| \left(\widehat{a}\widehat{a}_{ij}\frac{\partial\widehat{\widetilde{u}}_p}{\partial\widehat{x}_i}, \frac{\partial\widehat{w}}{\partial\widehat{x}_j}\right)_{\widehat{\Omega}} - \left(\widehat{a}\widehat{a}_{ij}\frac{\partial\widehat{\widetilde{u}}_p}{\partial\widehat{x}_i}, \frac{\partial\widehat{w}}{\partial\widehat{x}_j}\right)_{m,\widehat{\Omega}} | \\ &\le C\{ \|\Pi_r^n\widehat{a}\widehat{a}_{ij}\|_{0,\infty,\widehat{\Omega}} \|\frac{\partial\widehat{\widetilde{u}}_p}{\partial\widehat{x}_i} - \widehat{v}_q\|_{0,\widehat{\Omega}} \\ &\quad + \|\widehat{a}\widehat{a}_{ij} - \Pi_r^n\,\widehat{a}\widehat{a}_{ij}\|_{0,\infty,\widehat{\Omega}} \|\frac{\partial\widehat{\widetilde{u}}_p}{\partial\widehat{x}_i}\|_{0,\widehat{\Omega}}\,\} \|\frac{\partial\widehat{w}}{\partial\widehat{x}_j}\|_{0,\widehat{\Omega}}. \end{aligned}$$

Since $\|\Pi_r^n\,\widehat{a}\widehat{a}_{ij}\|_{0,\infty,\widehat{\Omega}} \le \|\widehat{a}\widehat{a}_{ij} - \Pi_r^n\,\widehat{a}\widehat{a}_{ij}\|_{0,\infty,\widehat{\Omega}} + \|\widehat{a}\widehat{a}_{ij}\|_{0,\infty,\widehat{\Omega}}$ we easily see from Lemma 2.3 and (1.10) that $\|\Pi_r^n\,\widehat{a}\widehat{a}_{ij}\|_{0,\infty,\widehat{\Omega}}$ is bounded by a fixed constant for any $r = d(m) - p - q > 0$. Moreover, taking $\widehat{v}_q = \Pi_q^1(\frac{\partial\widehat{\widetilde{u}}_p}{\partial\widehat{x}_i} + \widehat{\widetilde{u}}_p) + \Pi_q^1(\widehat{u} - \widehat{\widetilde{u}}_p) - \Pi_q^1\widehat{u}$ in $U_q(\widehat{\Omega})$ we have from Lemma 2.1 that

$$(3.51)\qquad \begin{aligned} &\|\Pi_r^n\widehat{a}\widehat{a}_{ij}\|_{0,\infty,\widehat{\Omega}} \|\frac{\partial\widehat{\widetilde{u}}_p}{\partial\widehat{x}_i} - \widehat{v}_q\|_{0,\widehat{\Omega}} \\ &\le C\,\{\|(\frac{\partial\widehat{\widetilde{u}}_p}{\partial\widehat{x}_i} + \widehat{\widetilde{u}}_p) - \Pi_q^1(\frac{\partial\widehat{\widetilde{u}}_p}{\partial\widehat{x}_i} + \widehat{\widetilde{u}}_p)\|_{0,\widehat{\Omega}} \\ &\quad + \|(\widehat{u} - \widehat{\widetilde{u}}_p) - \Pi_q^1(\widehat{u} - \widehat{\widetilde{u}}_p)\|_{0,\widehat{\Omega}} + \|\widehat{u} - \Pi_q^1\widehat{u}\|_{0,\widehat{\Omega}}\} \\ &\le C\,\{\varepsilon_q(\widehat{\widetilde{u}}_p) + q^{-1}\|\widehat{u} - \widehat{\widetilde{u}}_p\|_{1,\widehat{\Omega}} + q^{-\sigma}\|\widehat{u}\|_{\sigma,\widehat{\Omega}}\}, \end{aligned}$$

where $C$ is independent of $p$ and $q$.

In addition, we obtain from (3.35) that

Since $u_p - \widetilde{u}_p \in S_{p,0}(\Omega)$ the last term of the right side in (3.41) can be characterized as

$$(3.42) \qquad \|u_p - \widetilde{u}_p\|_{0,\Omega} = \sup_{w_p \in S_{p,0}(\Omega)} \frac{|(w_p, u_p - \widetilde{u}_p)_\Omega|}{\|w_p\|_{0,\Omega}}.$$

Hence we obtain from (3.40) that

$$(3.43) \qquad |(w_p, u_p - \widetilde{u}_p)_\Omega| = |B(w, u_p - \widetilde{u}_p)| \\ \leq |B(w, u_p) - B_{m,\Omega}(w, \widetilde{u}_p)| + |B_{m,\Omega}(w, \widetilde{u}_p) - B(w, \widetilde{u}_p)|.$$

Due to the fact that $B(\cdot,\cdot)$ is symmetric and $w \in S_{p,0}(\Omega)$, it follows from (1.16) and (2.5) that

$$(3.44) \qquad |(w, u_p - \widetilde{u}_p)_\Omega| \leq |B(\widetilde{u}_p, w) - B_{m,\Omega}(\widetilde{u}_p, w)| + |(f, w)_\Omega - (f, w)_{l,\Omega}|.$$

This completes the proof.

The above Lemma indicates that the error $\|u - \widetilde{u}_p\|_{0,\Omega}$ will depend on several terms. The first term $\|u - u_p\|_{0,\Omega}$ in (3.39) was already discussed in (1.20), which depends on the smoothness of the exact solution $u(x)$. The other terms will depend upon the smoothness of $a(x)$, $f(x)$ and the mapping $T$.

Now, for each $\widehat{t} \in U_p(\widehat{\Omega})$ we denote

$$(3.45) \qquad \varepsilon_q(\widehat{t}) = \max_i \left\| \left(\frac{\partial \widehat{t}}{\partial \widehat{x}_i} + \widehat{t}\right) - \Pi_q^1 \left(\frac{\partial \widehat{t}}{\partial \widehat{x}_i} + \widehat{t}\right) \right\|_{0,\widehat{\Omega}}, \qquad 0 < q \leq p.$$

Then, we obtain

$$(3.46) \qquad \varepsilon_q(\widehat{t}) \leq C\, q^{-(\lambda-1)} \|\widehat{t}\|_{\lambda,\widehat{\Omega}} \quad \text{for all} \quad \widehat{t} \in U_p(\widehat{\Omega}),$$

where $\lambda$ is a sufficiently large number. Moreover, it follows from (2.8) that

$$(3.47) \qquad \varepsilon_p(\widehat{t}) = 0 \quad \text{for all} \quad \widehat{t} \in U_p(\widehat{\Omega}).$$

Here,we have the following Proposition.

PROPOSITION 3.7. *Let $\widehat{u} \in H_0^\sigma(\widehat{\Omega})$, $\widehat{a} \in H^\alpha(\widehat{\Omega})$, $\widehat{J} \in H^\delta(\widehat{\Omega})$ and $\widehat{a}_{ij} \in H^\rho(\widehat{\Omega})$ for $i,j = 1,...,n$ with $k = min(\alpha, \rho) \geq n$. Then, for any $w \in S_{p,0}(\Omega)$ we have*

$$(3.48) \qquad |B(\widetilde{u}_p, w) - B_{m,\Omega}(\widetilde{u}_p, w)| \\ \leq C\,\{\varepsilon_q(\widehat{\widetilde{u}}_p) + q^{-\sigma}\|\widehat{u}\|_{\sigma,\widehat{\Omega}} + q^{-1}\|\widehat{u} - \widehat{\widetilde{u}}_p\|_{1,\widehat{\Omega}} \\ + r^{-(k-\frac{n}{2})}(\|\widehat{u} - \widehat{\widetilde{u}}_p\|_{1,\widehat{\Omega}} + \|\widehat{u}\|_{1,\widehat{\Omega}})\|\widehat{a}\|_{\alpha,\widehat{\Omega}} M_\rho\}\, \|\widehat{w}\|_{1,\widehat{\Omega}},$$

$$(3.52)\qquad \|\widehat{a}\widehat{a}_{ij} - \Pi_r^n \widehat{a}\widehat{a}_{ij}\|_{0,\infty,\widehat{\Omega}} \|\frac{\partial \widehat{\widetilde{u}}_p}{\partial \widehat{x}_i}\|_{0,\widehat{\Omega}} \leq C\, r^{-(k-\frac{n}{2})} \|\widehat{a}\|_{\alpha,\widehat{\Omega}} M_\rho \|\widehat{\widetilde{u}}_p\|_{1,\widehat{\Omega}} \leq C\, r^{-(k-\frac{n}{2})} \|\widehat{a}\|_{\alpha,\widehat{\Omega}} M_\rho (\|\widehat{u} - \widehat{\widetilde{u}}_p\|_{1,\widehat{\Omega}} + \|\widehat{u}\|_{1,\widehat{\Omega}}).$$

Thus, substituting (3.51) and (3.52) in (3.50) we complete the proof, since $\|\frac{\partial \widehat{w}}{\partial \widehat{x}_j}\|_{0,\widehat{\Omega}} \leq C\,\|\widehat{w}\|_{1,\widehat{\Omega}}$.

From Lemma 3.6,due to (3.2) and (3.48) we have the following theorem.

THEOREM 3.8. *For any $I_m$, $I_l \in G_p$, defined on $\widehat{\Omega} \subset R^n$, let $\widehat{u} \in H^\sigma(\widehat{\Omega})$, $\widehat{a} \in H^\alpha(\widehat{\Omega})$, $\widehat{J} \in H^\delta(\widehat{\Omega})$ $\widehat{f} \in H^\gamma(\widehat{\Omega})$ and $\widehat{a}_{ij} \in H^\rho(\widehat{\Omega})$ for $i, j = 1, \cdots, n$ such that $k = \min(\alpha, \rho, \gamma, \delta) \geq n$.Then, for any positive integers $q_1, q_2$ such that $0 < q_2 \leq d(l) - p - 1$ and $0 < q_1 \leq \min(d(m) - p - 1,\, p)$, we have*

$$(3.53)\qquad \|u - \widetilde{u}_p\|_{0,\Omega} \leq C\,\{\, {q_1}^{-\sigma}\|\widehat{u}\|_{\sigma,\widehat{\Omega}} + ({q_1}^{-1} + r_1^{-(k-\frac{n}{2})}\|\widehat{a}\|_{\alpha,\widehat{\Omega}}\, M_\rho)\, \|\widehat{u} - \widehat{\widetilde{u}}\|_{1,\widehat{\Omega}} + r_1^{-(k-\frac{n}{2})}\|\widehat{a}\|_{\alpha,\widehat{\Omega}}\, M_\rho\, \|\widehat{u}\|_{1,\widehat{\Omega}} + {q_2}^{-(\gamma-\frac{n}{2})}\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\, (\|\widehat{J}\|_{0,\infty,\widehat{\Omega}} + \|\widehat{J}\|_{\delta,\widehat{\Omega}}) + {r_2}^{-(\delta-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}}\, (\|\widehat{f}\|_{0,\infty,\widehat{\Omega}} + \|\widehat{f}\|_{\gamma,\widehat{\Omega}}) + \varepsilon_{q_1}(\widehat{\widetilde{u}}_p)\,\},$$

*where $k = \min(\alpha, \rho)$, $r_2 = d(l) - p - q_2$ and $r_1 = d(m) - p - q_1$.*

*proof.* For each $w_p \in S_{p,0}(\Omega)$ let $w \in S_{p,0}(\Omega)$ be the solution of (3.40). Then, since $w \in S_{p,0}(\Omega)$ we have $B(w, w) = |(w_p, w)_\Omega| \leq \|w_p\|_{0,\Omega}\|w\|_{0,\Omega}$. In addition, due to Poincaré's inequality and (1.10), we easily see that there exists a fixed constant $M$ such that

$$(3.54)\qquad \frac{\|\widehat{w}\|_{1,\widehat{\Omega}}}{\|w_p\|_{0,\Omega}} \leq M.$$

Thus, by a direct application of proposition 3.7 and Lemma 3.2 to

Lemma 3.6 we have

(3.55)

$$\sup_{w_p \in S_{p,0}(\Omega)} \frac{1}{\|w_p\|_{0,\Omega}} (|B(\widetilde{u}_p, w) - B_{m,\Omega}(\widetilde{u}_p, w)|$$
$$+ |(f,w)_\Omega - (f,w)_{l,\Omega}| \le C\{ {q_1}^{-\sigma} \|\widehat{u}\|_{\sigma,\widehat{\Omega}}$$
$$+ ({q_1}^{-1} + r_1^{-(k-\frac{n}{2})} \|\widehat{a}\|_{\alpha,\widehat{\Omega}} M_\rho) \|\widehat{u} - \widehat{\widetilde{u}}\|_{1,\widehat{\Omega}}$$
$$+ r_1^{-(k-\frac{n}{2})} \|\widehat{a}\|_{\alpha,\widehat{\Omega}} M_\rho \|\widehat{u}\|_{1,\widehat{\Omega}}$$
$$+ {q_2}^{-(\gamma-\frac{n}{2})} \|\widehat{f}\|_{\gamma,\widehat{\Omega}} (\|\widehat{J}\|_{0,\infty,\widehat{\Omega}} + \|\widehat{J}\|_{\delta,\widehat{\Omega}})$$
$$+ {r_2}^{-(\delta-\frac{n}{2})} \|\widehat{J}\|_{\delta,\widehat{\Omega}} (\|\widehat{f}\|_{0,\infty,\widehat{\Omega}} + \|\widehat{f}\|_{\gamma,\widehat{\Omega}}) + \varepsilon_{q_1}(\widehat{\widetilde{u}}_p)\}.$$

Moreover, it follows from (1.20) that the first term of the right side in (3.39) is dominated by the first term in (3.55). This completes the proof.

When $d(m)$ and $d(l)$ are large enough with $q_1 = q_2 = p$, the rate of convergence for $\|\widehat{u} - \widehat{\widetilde{u}}_p\|_{1,\Omega}$ is asymptotically $O(p^{-(\sigma-1)})$, which coincides with that of $\|u - u_p\|_{1,\Omega}$. Also, it follows from (3.47) that the $L_2(\Omega)$ error $\|u - \widetilde{u}_p\|_{0,\Omega}$ in (3.53) is asymptotically $O(p^{-\sigma})$ under nearly exact integrations, which is the same with that of $\|u - u_p\|_{0,\Omega}$ in (1.20). Moreover, we see that under certain conditions the $L_2(\Omega)$ error $\|u - \widetilde{u}_p\|_{0,\Omega}$ has nearly $O(p^{-1})$ improvement over the $H^1$ error $\|u - \widetilde{u}_p\|_{1,\Omega}$. In the case where $a$ and $f$ are sufficiently smooth, i.e., $\alpha$ and $\gamma$ are large enough, even when $d(m) \approx 2p+1$ with $q_1 = p$ and $d(l) \approx p+1$ the first term of the right side in (3.53) may dominate the other terms, so that the rate of convergence for $\|u - \widetilde{u}_p\|_{0,\Omega}$ is asymptotically $O(p^{-\sigma})$. When $a$ or $f$ is not smooth enough we may reduce the error $\|u - \widetilde{u}_p\|_{0,\Omega}$ by increasing the value of $d(m)$ or $d(l)$ respectively. In fact, using overintegrations $I_m(m > p)$ or $I_l(l > p)$ we recover the optimal rate of convergence for $\|u - \widetilde{u}_p\|_{0,\Omega}$.

## 4. Numerical experiments

We consider the following one-dimensional problem:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) = f \quad \text{on} \quad \Omega = [0,1]$$

with $u(0) = u(1) = 0$.
Here, $a$ and $f$ are chosen in such a way that the exact solution is $u(x) = e^x \sin(x) - e^1 \sin(1)\, x$. Of course, the simulations have no need for the knowledge of the exact solution $u$.

EXAMPLE 4.1. We choose $a(x) = 1/(x+w)$ for $w > 0$ and take the mapping $T(\widehat{x}) = ((2+\varepsilon)^\alpha - (1-\widehat{x}+\varepsilon)^\alpha)/((2+\varepsilon)^\alpha - \varepsilon^\alpha)$ with $\alpha = 2.5$ and $\varepsilon = 0.001$, If $w$ is near to zero, then $a(x)$ and $f(x)$ have poles near to $x = 0$ in the original problem. Hence we need the over-integrations $L_m$ and $L_l$ in both of the stiffness matrix and the load vector. When we choose $w = 0.001$, the $H^1(\Omega)$ and $L_2(\Omega)$-error results in Figure 4.1.1 and 4.1.2 respectively, follow under the case where $L_m(m = 1000)$ and $L_l(l \geq p)$.

We consider the following two-dimensional problem:

$$-\text{div}(a\nabla u) = f \quad \text{on} \quad \Omega \subset R^2, \text{ with } u(x) = 0 \text{ on } \Gamma.$$

EXAMPLE 4.2. In the case where the domain $\Omega$ is the trapezoid with vertices $A = (0,0)$, $B = (2,0)$, $C = (0,1)$, $D = (1,1)$, we consider mapping $T : (\widehat{x}_1, \widehat{x}_2) \in \widehat{\Omega} \longrightarrow (x_1, x_2) \in \Omega$ given by $x_1 = (\widehat{x}_1+1)(3-\widehat{x}_2)/4$, $x_2 = (\widehat{x}_2+1)/2$. We choose $a(x_1,x_2)$, $f(x_1,x_2)$ in such a way that $u(x_1,x_2) = x_1x_2(x_1+x_2-2)(e^{(x_2-1)}-1)$. In particular, we take $a(x_1,x_2) = 1/(x_1+w)$ with $w > 0$. If $w$ is near to zero, then $a(x_1,x_2)$ has a singularity near to the $x_2$-axis, and also $f$ is singular. Hence, even if the mapping $T$ is smooth enough, $\widehat{a}\,\widehat{a}_{ij}$ and $\widehat{J}\widehat{f}$ are not sufficiently smooth, which is caused by the original problem. To obtain optimal results we may use overintegrations $L_m$ and $L_l$. When $w = 0.05$, Figure 4.2.1 and 4.2.2 show the results in the case where $L_m(m = 50)$ and $L_l(l \geq p+1)$ are used.

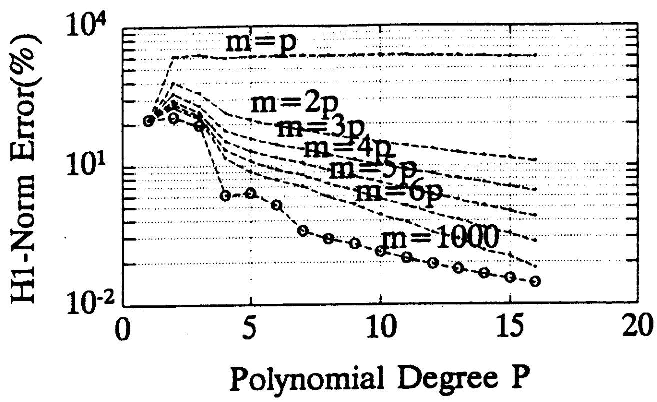


Figure 4.1.1

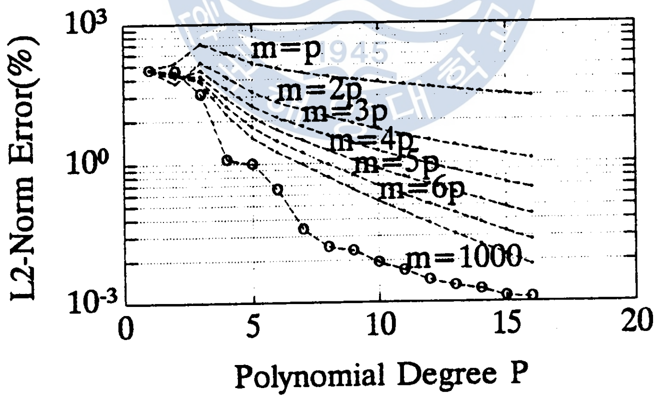


Figure 4.1.2

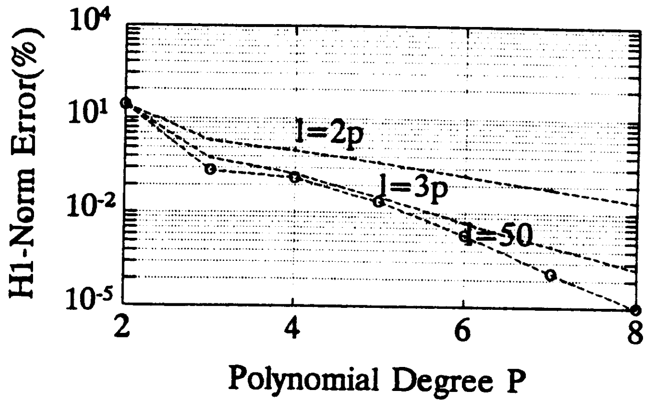


Figure 4.2.1

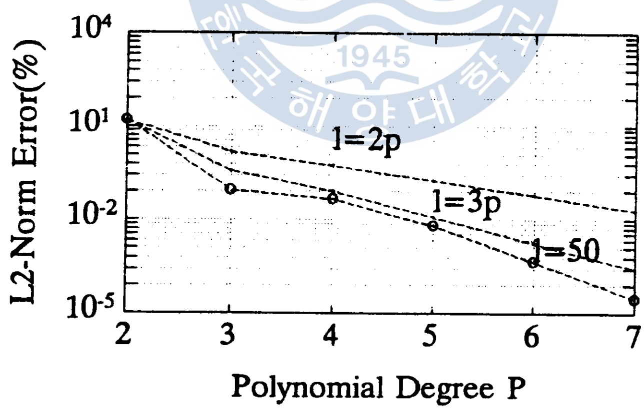


Figure 4.2.2

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Department of Applied Mathematics
Korea Maritime University
Pusan 606-791, Korea