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On Certain Class of Meromorphic functions with Positive Coefficients

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1. Introduction

Let $\sum_p$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n, \qquad (a_n \geq 0;\ p \in N = \{1, 2, \cdots\}) \tag{1.1}$$

which are analytic and univalent in $D = \{z : 0 < |z| < 1\}$ with a simple pole at the origin with residue 1 there.

A function $f(z)$ in $\sum_p$ is said to be a member of $\sum_p(A, B)$ if it satisfies

$$\left|\frac{zf'(z)}{f(z)} + 1\right| < \left|A + B\frac{zf'(z)}{f(z)}\right| \tag{1.2}$$

for $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $z \in D$.

In particular, the class $\sum_1(-1, 1)$ was studied by Padmanabhan [5] and the class $\sum_1(A, B)$ when $A = \beta(2\alpha - I)$ and $B = \beta$ $(0 \leq \alpha < 1$ and $0 < \beta \leq 1)$ were studied by Mogra, Reddy, and Juneja [4].

The aim of the present paper is to investigate coefficient estimates, distortion properties and radius of convexity for the class $\sum_p(A,B)$. Furthermore, it is shown that the class $\sum_p(A,B)$ is closed under convex linear combinations, convolutions and integral transforms.

2. Coefficient estimates

Theorem 1. Let

$$f(z)=\frac{1}{z}+\sum_{n=p}^{\infty}a_nz^n, \qquad (a_n\geq 0),$$

be regular in D. Then $f(z)$ is in the class $\sum_p(A,B)$ if and only if

$$\sum_{n=p}^{\infty}\{(n+1)+(A+Bn)\}a_n\leq B-A, \tag{2.1}$$

for $-1\leq A<B\leq 1$ and $0<B\leq 1$.

Proof. Suppose that

$$f(z)=\frac{1}{z}+\sum_{n=p}^{\infty}a_nz^n, \qquad (a_n\geq 0),$$

is in $\sum_p(A,B)$, Then

$$\left|\frac{\frac{zf'(z)}{f(z)}+1}{A+B\frac{zf'(z)}{f(z)}}\right|=\left|\frac{\sum_{n=p}^{\infty}(n+1)a_nz^n}{(B-A)\frac{1}{z}-\sum_{n=p}^{\infty}(A+Bn)a_nz^n}\right|<1$$

for all $z \in D$. Since $\mathrm{Re}(z) \leq |z|$ for all z, we have

$$\mathrm{Re}\left\{ \frac{\sum_{n=p}^{\infty}(n+1)a_n z^n}{(B-A)\frac{1}{z} - \sum_{n=p}^{\infty}(A+Bn)a_n z^n} \right\} < 1, \quad (z \in D). \tag{2.2}$$

Now choose the values of z on real axis so that $\dfrac{zf'(z)}{f(z)}$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1$ through positive values, we obtain

$$\sum_{n=p}^{\infty}\{(n+1)+(A+Bn)\}a_n \leq B-A.$$

Conversely, suppose that (2.1) holds for all admissible values of A and B. Then we have

$$\begin{aligned} H(f,f') &= |zf'(z)+f(z)| - |Af(z)+Bzf'(z)| \\ &= \left|\sum_{n=p}^{\infty}(n+1)a_n z^n\right| - \left|(B-A)\frac{1}{z} - \sum_{n=p}^{\infty}(A+Bn)a_n z^n\right| \end{aligned}$$

or

$$\begin{aligned} &|z|H(f,f') \\ &\leq \sum_{n=p}^{\infty}(n+1)a_n|z|^{n+1} - (B-A) + \sum_{n=p}^{\infty}(A+Bn)a_n|z|^{n+1} \\ &= \sum_{n=p}^{\infty}\{(n+1)+(A+Bn)\}a_n|z|^{n+1} - (B-A). \end{aligned}$$

Since the above inequality holds for all $r = |z|$, $0 < r < 1$, letting $r \to 1$, we have

$$\sum_{n=p}^{\infty}\{(n+1)+(A+Bn)\}a_n \leq B - A$$

by (2.1). Hence it follows that $f(z)$ is in the class $\sum_p(A, B)$.

Corollary. If the function

$$f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$$

is in the class $\sum_p(A, B)$, then we have

(2.3) $$a_n \leq \frac{B-A}{(n+1)+(A+Bn)}, \quad (n \geq p).$$

The result is sharp for the function

(2.4) $$f_n(z) = \frac{1}{z} + \frac{B-A}{(n+1)+(A+Bn)}z^n, \qquad (n \geq p).$$

3. Distortion properties and radius of convexity

Theorem 2. If the function

$$f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$$

is in the class $\sum_p(A, B)$, then we have

$$\frac{1}{|z|} - \frac{B-A}{(p+1)+A+Bp}|z|^p$$

$$\leq |f(z)| \leq \frac{1}{|z|} + \frac{B-A}{(p+1)+A+Bp}|z|^p.$$

The result is sharp.

Proof. Suppose that $f(z)$ is in $\sum_p(A,B)$. By Theorem 1, we have

$$\sum_{n=p}^{\infty} a_n \leq \frac{B-A}{(p+1)+A+Bp}.$$

Thus

$$\begin{aligned}|f(z)| &\leq \frac{1}{|z|} + |z|^p \sum_{n=p}^{\infty} a_n \\ &\leq \frac{1}{|z|} + \frac{B-A}{(p+1)+A+Bp}|z|^p.\end{aligned}$$

Also,

$$\begin{aligned}|f(z)| &\geq \frac{1}{|z|} - |z|^p \sum_{n=p}^{\infty} a_n \\ &\geq \frac{1}{|z|} - \frac{B-A}{(p+1)+A+Bp}|z|^p.\end{aligned}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{B-A}{(p+1)+A+Bp} z^p.$$

Theorem 3. If the function

$$f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$$

is in the class $\sum_p(A,B)$, then $f(z)$ is meromorphically convex of order δ $(0 \leq \delta < 1)$ in $|z| < r = r(A,B,\delta)$, where

$$r(A,B,\delta) = \inf_{n \geq p} \left\{ \frac{(1-\delta)(n+1)+(A+Bn)}{(B-A)n(n+2-\delta)} \right\}^{\frac{1}{n+1}}.$$

The result is sharp.

Proof. Let $f(z)$ is in $\sum_p^*(A,B)$. Then, by Theorem 1, we have

$$\sum_{n=p}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} a_n \leq 1. \tag{3.1}$$

It is sufficient to show that

$$\left| 2+\frac{2f''(z)}{f'(z)} \right| \leq 1-\delta$$

for $|z| \leq r(A,B,\delta)$, where $r(A,B,\delta)$ is as specified in the statement of the theorem. Then

$$\left| 2+\frac{zf''(z)}{f'(z)} \right| = \left| \frac{\sum_{n=p}^{\infty} n(n+1)a_n z^{n-1}}{-\frac{1}{z^2} - \sum_{n=p}^{\infty} na_n z^{n-1}} \right|$$

$$\leq \frac{\sum_{n=p}^{\infty} n(n+1)a_n |z|^{n+1}}{1-\sum_{n=p}^{\infty} na_n |z|^{n+1}}.$$

This will be bounded by $1-\delta$ if

$$\sum_{n=p}^{\infty} \frac{n(n+2-\delta)}{1-\delta} a_n |z|^{n+1} \leq 1. \tag{3.2}$$

By (3.1), it follows that (3.2) is true if

$$\frac{n(n+2-\delta)}{1-\delta}|z|^{n+1} \leq \frac{(n+1)+(A+Bn)}{B-A}, \qquad (n \geq p)$$

or

$$(3.3) \qquad |z| \leq \left\{ \frac{(1-\delta)\{(n+1)+(A+Bn)\}}{(B-A)n(n+2-\delta)} \right\}^{\frac{1}{n+1}}, \qquad (n \geq p).$$

Setting $|z| = r(A, B, \delta)$ in (3.3), the result follows. The result is sharp for the function

$$(3.4) \qquad f_n(z) = \frac{1}{z} + + \frac{B-A}{(n+1)+(A+Bn)} z^n, \qquad (n \geq p).$$

Remark. A function $f(z) \in \sum_p$ is said to be meromorphically convex of order δ $(0 \leq \delta < 1)$ if

$$\mathrm{Re}\left\{ -\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \delta, \qquad (z \in D).$$

4. Convex linear combinations and convolution properties

Theorem 4. Let $f_0(z) = \dfrac{1}{z}$ and

$$f_n(z) = \frac{1}{z} + \frac{B-A}{(n+1)+(A+Bn)} z^n, \qquad (n \geq p).$$

Then

$$f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$$

is in the class $\sum_p(A,B)$ if and only if it can be expressed in the form

$$f(z) = \lambda_0 f_0(z) + \sum_{n=p}^{\infty} \lambda_n f_n(z),$$

where $\lambda_0 \geq 0$, $\lambda_n \geq 0$ $(n \geq p)$ and $\lambda_0 + \sum_{n=p}^{\infty} \lambda_n = 1$.

Proof. Let

$$f(z) = \lambda_0 f_0(z) + \sum_{n=p}^{\infty} \lambda_n f_n(z),$$

with $\lambda_0 \geq 0$, $\lambda_n \geq 0$ $(n \geq p)$ and $\lambda_0 + \sum_{n=p}^{\infty} \lambda_n = 1$. Then

$$\begin{aligned} f(z) =& \lambda_0 f_0(z) + \sum_{n=p}^{\infty} \lambda_n f_n(z) \\ =& \frac{1}{z} + \sum_{n=p}^{\infty} \lambda_n \frac{B-A}{(n+1)+(A+Bn)} z^n. \end{aligned}$$

Since

$$\begin{aligned} \sum_{n=p}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} \lambda_n \frac{B-A}{(n+1)+(A+Bn)} =& \sum_{n=p}^{\infty} \lambda_n \\ =& 1 - \lambda_0 \leq 1, \end{aligned}$$

by Theorem 1, $f(z)$ is in the class $\sum_p(A,B)$.

Conversely, suppose that the function $f(z)$ is in the class $\sum_p(A,B)$. Since

$$a_n \leq \frac{B-A}{(n+1)+(A+Bn)}, \qquad (n \geq p),$$

setting

$$\lambda_n = \frac{(n+1)+(A+Bn)}{B-A} a_n, \qquad (n \geq p),$$

and

$$\lambda_0 = 1 - \sum_{n=p}^{\infty} \lambda_n,$$

it follows that

$$f(z) = \lambda_0 f_0(z) + \sum_{n=p}^{\infty} \lambda_n f_n(z).$$

This completes the proof of the theorem.

For the function $f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=p}^{\infty} b_n z^n$ belonging to $\sum_p$, we denote by $(f * g)(z)$ the convolution of $f(z)$ and $g(z)$, or

$$(f * g)(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n b_n z^n.$$

Theorem 5. If the function $f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=p}^{\infty} b_n z^n$ are in the class $\sum_p(A, B)$, then

$$(f * g)(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n b_n z^n.$$

is in the class $\sum_p(A, B)$.

Proof. Suppose that $f(z)$ and $g(z)$ are in $\sum_p(A,B)$. By Theorem 1, we have

$$\sum_{n=p}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} a_n \leq 1$$

and

$$\sum_{n=p}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} b_n \leq 1.$$

Since $f(z)$ and $g(z)$ are regular in D, so is $(f*g)(z)$. Furthermore,

$$\begin{aligned}
&\sum_{n=p}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} a_n b_n \\
\leq &\sum_{n=p}^{\infty} \left\{\frac{(n+1)+(A+Bn)}{B-A}\right\}^2 a_n b_n \\
\leq &\left[\sum_{n=p}^{\infty} \left\{\frac{(n+1)+(A+Bn)}{B-A}\right\} a_n\right] \left[\sum_{n=p}^{\infty} \left\{\frac{(n+1)+(A+Bn)}{B-A}\right\} b_n\right] \\
\leq &1.
\end{aligned}$$

Hence by Theorem 1, $(f*g)(z)$ is in the class $\sum_p(A,B)$.

5. Integral transforms

In this section, we consider integral transforms of functions in $\sum_p(A,B)$ of the type considered by Bajpai [1] and Goel and Sohi [3].

Theorem 6. If the function $f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$ is in the class $\sum_p(A,B)$, then the integral transforms

$$F_c(z) = c\int_0^1 u^c f(uz)du, \qquad (0 < c < \infty)$$

are in the class $\sum_p(A,B)$.

Proof. Suppose that $f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$ is in $\sum_p(A,B)$. Then we have

$$F_c(z) = c\int_0^1 u^c f(uz)du = \frac{1}{z} + \sum_{n=p}^{\infty} \frac{ca_n}{n+c+1} z^n.$$

Since

$$\sum_{n=p}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} \frac{ca_n}{n+c+1}$$
$$\leq \sum_{n=p}^{\infty} \frac{(n+1)+(A+Bn)}{B-A} a_n$$
$$\leq 1$$

by Theorem 1, it follows that $F_c(z)$ is in the class $\sum_p(A,B)$.

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