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On Certain Classes of Multivalent Functions with Negative Coefficients

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Abstract

In the present paper, we obtain sharp results concerning coefficient estimates distortion theorem, closure theorems and radius of convexity for the class $S^*(p, n, \lambda, A, B)$. We also obtain class preserving integral operators of the form

$$F(z) = \frac{p+c}{z^c} \int_0^z t^{c-1} f(t)dt \qquad (c > -p)$$

for the class $S^*(p, n, \lambda, A, B)$. Also we determine radius of p-valence of $f(z)$ when $F(z) \in S^*(p, n, \lambda, A, B)$. Furthermore we obtain distortion theorem for the fractional integral.

1. Introduction

Let A_p denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \qquad (p \in N = \{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

Let $f(z)$ be in A_p and $g(z)$ be in A_p. Then we denote by $f * g(z)$ the Hadamard product or convolution of $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p} \qquad (p \in N),$$

then

$$f * g(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p}.$$

Let

$$D^{n+p-1} f(z) = \left(\frac{z^p}{(1-z)^{n+p}} \right) * f(z) = \frac{z^p (n^{n-1} f(z))^{n+p-1}}{(n+p-1)!},$$

where n is any integer greater than $-p$.
Particularly, the symbol $D^n f(z)$ was named the n-th order Ruscheweyh derivative of $f(z)$ by Al-Amir [1]. Recently, some classes defined by using the symbol $D^{n+p-1} f(z)$ were studied by Goel and Sohi [4], Sohi [9] and Owa [6, 7].

Now we introduce the following classes by using the symbol $D^{n+p-1} f(z)$.

For $\lambda \geq 0$, $-1 \leq A < B \leq 1$ and $n > -p$, let $S(p, n, \lambda, A, B)$ be the class of functions $f(z)$ of A_p for which

$$(1-\lambda) \frac{D^{n+p-1} f(z)}{z^p} + \lambda \frac{D^{n+p} f(z)}{z^p}$$

is subordinate to $(1+Az)/(1+Bz)$. In other words, $f(z) \in S(p,n,\lambda,A,B)$ if and only if there exists a function $w(z)$ analytic in U and satisfying $w(0)=0$, $|w(z)|<1$ for $z \in U$, such that

$$(1-\lambda)\frac{D^{n+p-1}f(z)}{z^p}+\lambda\frac{D^{n+p}f(z)}{z^p}=\frac{1+Aw(z)}{1+Bw(z)}.$$

Let T_p denote the subclass of A_p consisting of functions of the form

$$f(z)=z^p-\sum_{k=1}^{\infty}a_{k+p}z^{k+p} \qquad (a_{k+p}\geq 0).$$

We denote by $S^*(p,n,\lambda,A,B)$ the class obtained by taking intersection of the class $S(p,n,\lambda,A,B)$ with T_p.

The classes $S^*(1,0,\lambda,2a-1,1)$ with $0 \leq a < 1$ and $S^*(1,n,\lambda,A,B)$ have been studied by Bhoosurmath and Swamy [2] and Chen, Yu and Owa [3], respectively.

2. Coefficient estimates

Theorem 1. A function

$$f(z)=z^p-\sum_{k=1}^{\infty}a_{k+p}z^{k+p} \qquad (a_{k+p}\geq 0)$$

is in the class $S^*(p,n,\lambda,A,B)$ if and only if

$$\sum_{k=1}^{\infty}\frac{(k+p+n-1)!(n+p+\lambda k)}{(n+p)!k!}a_{k+p}\leq\frac{B-A}{1+B}, \tag{2.1}$$

where $\lambda\geq 0$, $1\leq A<B\leq 1$, $0<B\leq 1$ and $n>-p$. The result is sharp.

Proof. Suppose that $f(z) \in S^*(p,n,\lambda,A,B)$. Then we have

$$h(z) = (1-\lambda)\frac{D^{n+p-1}f(z)}{z^p} + \lambda\frac{D^{n+p}f(z)}{z^p} = \frac{1+Aw(z)}{1+Bw(z)}$$

for $\lambda > 0$, $-1 \le A < B \le 1$, $0 < B \le 1$, $z \in U$ and $w(z) \in H = \{w(z)$ analytic, $w(0)=0$ and $|w(z)| < 1$ for $z \in U\}$. From this we get

$$w(z) = \frac{1-h(z)}{Bh(z)-A}.$$

Since

$$D^{n+p-1}f(z) = \frac{z^p(z^{n-1}f(z))^{n+p-1}}{(n+p-1)!} = z^p - \sum_{k=1}^{\infty}\frac{(k+p+n-1)!}{(n+p-1)!k!}a_{k+p}z^{k+p},$$

therefore

$$h(z) = 1 - \sum_{k=1}^{\infty}\frac{(k+p+n-1)!(n+p+\lambda k)}{(n+p)!k!}a_{k+p}z^k$$

and $|w(z)| < 1$ implies

$$\left|\frac{\sum_{k=1}^{\infty}\frac{(k+p+n-1)!(n+p+\lambda k)}{(n+p)!k!}a_{k+p}z^k}{(B-A) - B\sum_{k=1}^{\infty}\frac{(k+p+n-1)!(n+p+\lambda k)}{(n+p)!k!}a_{k+p}z^k}\right| < 1. \tag{2.2}$$

Since $|\mathrm{Re}(z)| \le |z|$ for all z, we have

$$\mathrm{Re}\left|\frac{\sum_{k=1}^{\infty}\frac{(k+p+n-1)!(n+p+\lambda k)}{(n+p)!k!}a_{k+p}z^k}{(B-A) - B\sum_{k=1}^{\infty}\frac{(k+p+n-1)!(n+p+\lambda k)}{(n+p)!k!}a_{k+p}z^k}\right| < 1. \tag{2.3}$$

We consider real values of z and take $0 \leq r = |z| < 1$. Then, for $r = 0$, the denominator of (2.3) is positive and so it is positive for all $0 \leq r < 1$, since $w(z)$ is analytic for $|z| < 1$. Then (2.3) gives

$$\sum_{k=1}^{\infty} \frac{(k+p+n-1)!(n+p+\lambda k)}{(n+p)!k!} a_{k+p} r^k \leq \frac{B-A}{1+B}. \tag{2.4}$$

Letting $r \to 1$, we obtain (2.1).

Conversely, suppose that $f(z) \in T_p$ and satisfies (2.1). For $|z| = r$, $0 \leq r < 1$, we have (2.4) by (2.1), since $r^k < 1$. So we have

$$\begin{aligned} &\left| \sum_{k=1}^{\infty} \frac{(k+p+n-1)!(n+p+\lambda k)}{(n+p)!k!} a_{k+p} z^k \right| \\ \leq &\sum_{k=1}^{\infty} \frac{(k+p+n-1)!(n+p+\lambda k)}{(n+p)!k!} a_{k+p} r^k \\ < &(B-A) - B \sum_{k=1}^{\infty} \frac{(k+p+n-1)!(n+p+\lambda k)}{(n+p)!k!} a_{k+p} r^k \\ < &\left| (B-A) - B \sum_{k=1}^{\infty} \frac{(k+p+n-1)!(n+p+\lambda k)}{(n+p)!k!} a_{k+p} z^k \right|, \end{aligned}$$

which gives (2.2) and hence follows that

$$(1-\lambda)\frac{D^{n+p-1}f(z)}{z^p} + \lambda \frac{D^{n+p}f(z)}{z^p} = \frac{1+Aw(z)}{1+Bw(z)},$$

for $\lambda \geq 0$, $-1 \leq A < B \leq 1$, $z \in U$ and $w(z) \in H$. That is, $f(z) \in S^*(p, n, \lambda, A, B)$. The function

$$f(z) = z^p - \frac{(n+p)!k!(B-A)}{(k+p+n-1)!(n+p+\lambda k)(1+B)} z^{k+p} \qquad (k \in N) \tag{2.5}$$

is an extremal function.

Corollary 1. If a function

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \qquad (a_{k+p} \geq 0)$$

is in the class $S^*(p, n, \lambda, A, B)$, then

$$a_{k+p} \leq \frac{(n+p)!k!(B-A)}{(k+p+n-1)!(n+p+\lambda k)(1+B)} \qquad (k \in N).$$

The equality holds for the functions given by (2.5).

3. Distortion theorem

Theorem 2. If $f(z) \in S^*(p, n, \lambda, A, B)$, then

$$r^p - \frac{B-A}{(n+p+\lambda)(1+B)} r^{p+1} \leq |f(z)|$$

$$\leq r^p + \frac{B-A}{(n+p+\lambda)(1+B)} r^{p+1} \qquad (|z| = r), \tag{3.1}$$

and, for $\lambda \geq (n+p)/p$

$$pr^{p-1} - \frac{(p+1)(B-A)}{(n+p+\lambda)(1+B)} r^p \leq |f'(z)|$$

$$\leq pr^{p-1} + \frac{(p+1)(B-A)}{(n+p+\lambda)(1+B)} r^p \qquad (|z| = r). \tag{3.2}$$

The results are sharp.

Proof. Since $\frac{(k+p+n-1)!}{k!}$ is an increasing function of k, we have, from Theorem 1,

$$\sum_{k=1}^{\infty} a_{k+p} \leq \frac{B-A}{(n+p+\lambda)(1+B)}. \tag{3.3}$$

Hence

$$|f(z)| \leq |z|^p + \sum_{k=1}^{\infty} a_{k+p}|z|^{k+p} \leq r^p + r^{p+1}\sum_{k=1}^{\infty} a_{k+p}$$

$$\leq r^p + \frac{B-A}{(n+p+\lambda)(1+B)} r^{p+1} \qquad (|z|=r).$$

Similarly,

$$|f(z)| \geq |z|^p - \sum_{k=1}^{\infty} a_{k+p}|z|^{k+p}$$

$$\geq r^p - \frac{B-A}{(n+p+\lambda)(1+B)} r^{p+1} \qquad (|z|=r).$$

Thus (3.1) follows. Also, in view of the inequality (2.1) and (3.3), we have

$$\sum_{k=1}^{\infty}(k+p)a_{k+p} \leq \frac{1}{\lambda}\left(\frac{B-A}{1+B} - (n+p-\lambda p)\sum_{k=1}^{\infty} a_{k+p}\right)$$

$$\leq \frac{1}{\lambda}\left(\frac{B-A}{1+B}\left(1 - \frac{n+p-\lambda p}{n+p+\lambda}\right)\right)$$

$$= \frac{(p+1)(B-A)}{(n+p+\lambda)(1+B)}.$$

This implies that

$$|f'(z)| \leq p|z|^{p-1} + \sum_{k=1}^{\infty}(k+p)a_{k+p}|z|^{k+p-1}$$

$$\leq pr^{p-1} + r^p \sum_{k=1}^{\infty}(k+p)a_{k+p}$$

$$< pr^{p-1} + \frac{(p+1)(B-A)}{(n+p+\lambda)(1+B)}r^p \qquad (|z|=r).$$

Similarly,

$$|f'(z)| \geq p|z|^{p-1} - \sum_{k=1}^{\infty}(k+p)a_{k+p}|z|^{k+p-1}$$

$$\geq pr^{p-1} - \frac{(p+1)(B-A)}{(n+p+\lambda)(1+B)}r^p \qquad (|z|=r).$$

The bounds are sharp for the function

$$f(z) = z^p - \frac{B-A}{(n+p+\lambda)(1+B)}z^{p+1}. \tag{3.4}$$

4. Closure theorems

Theorem 3. Let

$$f_i(z) = z^p - \sum_{k=1}^{\infty} a_{i,k+p} z^{k+p} \qquad (a_{i,k+p} \geq 0)$$

is in the class $S^*(p,n,\lambda,A,B)$ for each $i = 1, 2, \cdots, m$. Then the function

$$h(z) = z^p - \frac{1}{m}\sum_{k=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{i,k+p}\right) z^{k+p}$$

is in the class $S^*(p,n,\lambda,A,B)$.

Proof. Since $f_i(z) \in S^*(p,n,\lambda,A,B)$ for each $i = 1,2,\cdots,m$, we have

$$\sum_{k=1}^{\infty} \frac{(k+p+n-1)!(n+p+\lambda k)}{(n+p)!k!} a_{i,k+p} < \frac{B-A}{1+B}$$

by Theorem 1. Hence we obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{(k+p+n-1)!(n+p+\lambda k)}{(n+p)!k!} \left(\frac{1}{m} \sum_{k=1}^{m} a_{i,k+p} \right) \\
&= \frac{1}{m} \sum_{k=1}^{\infty} \left[\sum_{k=1}^{m} \frac{(k+p+n-1)!(n+p+\lambda k)}{(n+p)!k!} a_{i,k+p} \right] \\
&\leq \frac{1}{m} \sum_{k=1}^{m} \frac{B-A}{1+B} = \frac{B-A}{1+B},
\end{aligned}$$

which shows that $h(z) \in S^*(p,n,\lambda,A,B)$.

Theorem 4. Let $f_p(z) = z^p$ and

$$f_{k+p}(z) = z^p - \frac{(n+p)!k!(B-A)}{(k+p+n-1)!(n+p+\lambda k)(1+B)} z^{k+p} \qquad (k \in N).$$

Then $f(z) \in S^*(p,n,\lambda,A,B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \lambda_{k+p} f_{k+p}(z),$$

where $\lambda_{k+p} \geq 0$ and $\sum_{k=0}^{\infty} \lambda_{k+p} = 1$.

Proof. Suppose that

$$f(z)=\sum_{k=0}^{\infty}\lambda_{k+p}f_{k+p}(z)$$

$$=z^p-\sum_{k=0}^{\infty}\frac{(n+p)!k!(B-A)}{(k+p+n-1)!(n+p+\lambda k)(1+B)}\lambda_{k+p}z^{k+p}.$$

Then

$$\sum_{k=1}^{\infty}\frac{(k+p+n-1)!(n+p+\lambda k)(1+B)}{(n+p)!k!(B-A)}$$

$$\lambda_{k+p}\frac{(n+p)!k!(B-A)}{(k+p+n-1)!(n+p+\lambda k)(1+B)}$$

$$=\sum_{k=1}^{\infty}\lambda_{k+p}=1-\lambda_p\leq 1.$$

Hence, by Theorem 1, $f(z)\in S^*(p,n,\lambda,A,B)$.

Conversely, suppose that $f(z)\in S^*(p,n,\lambda,A,B)$. Since

$$a_{k+p}\leq\frac{(n+p)!k!(B-A)}{(k+p+n-1)!(n+p+\lambda k)(1+B)}\qquad(k\in N),$$

we may set

$$\lambda_{k+p}=\frac{(k+p+n-1)!(n+p+\lambda k)(1+B)}{(n+p)!k!(B-A)}a_{k+p}\qquad(k\in N)$$

and

$$\lambda_p=1-\sum_{k=1}^{\infty}\lambda_{k+p}.$$

Then

$$f(z) = \sum_{k=0}^{\infty} \lambda_{k+p} f_{k+p}(z).$$

This completes the proof of the theorem.

5. Radius of convexity for the class $S^*(p, n, \lambda, A, B)$

Theorem 5. If $f(z) \in S^*(p, n, \lambda, A, B)$, then $f(z)$ is p-valent for $|z| < r_p$, where

$$r_p = \inf_k \left[\frac{(k+p+n-1)!(n+p\lambda+k(1+B)p)}{(n+p)!k!(B-A)(k+p)} \right]^{\frac{1}{k}} . \qquad (k \in N).$$

The result is sharp.

Proof. It is sufficient to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| < p$$

for $|z| < r_p$. Now

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=1}^{\infty} (k+p) a_{k+p} |z|^k.$$

Thus

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| < p$$

if

$$\sum_{k=1}^{\infty} \left(\frac{k+p}{p} \right) a_{k+p} |z|^k < 1. \qquad (5.1)$$

But Theorem 1 confirms that

$$\sum_{k=1}^{\infty} \frac{(k+p+n-1)!(n+p+\lambda k)(1+B)}{(n+p)!k!(B-A)} a_{k+p} \leq 1.$$

Thus (5.1) will be satisfied if

$$\binom{k+p}{p} a_{k+p}|z|^k \leq \frac{(k+p+n-1)!(n+p+\lambda k)(1+B)}{(n+p)!k!(B-A)} a_{k+p} \qquad (k \in N),$$

or if

$$|z| \leq \left[\frac{(k+p+n-1)!(n+p+\lambda k)(1+B)p}{(n+p)!k!(B-A)(k+p)}\right]^{\frac{1}{k}}. \qquad (5.2)$$

The required result follows now from (5.2). The result is sharp for the function given by (2.5).

By using the similar method of the proof in Theorem 5, we have

Theorem 6. If $f(z) \in S^*(p,n,\lambda,A,B)$, then $f(z)$ is p-valently convex in the disk

$$|z| < r_p^* = \inf_k \left[\frac{(k+p+n-1)!(n+p+\lambda k)(1+B)p^2}{(n+p)!k!(B-A)(k+p)^2}\right]^{\frac{1}{p}} \qquad (k \in N).$$

The result is sharp for the function given by (2.5)..

Remark. (1) Putting $p=1$, $n=0$, $B=1$ and $A=2a-1$ $(0 \leq a < 1)$ in the above theorem, we get the results obtained by Bhoosnurmath and Swamy [2].

(2) Putting $p=1$ in the above theorems, we get the results obtained by Chen, Yu and Owa [3].

6. Integral operators

Theorem 7. Let c be a real number such that $c > -p$. If $f(z) \in S^*(p,n,\lambda,A,B)$, then the function $F(z)$ defined by

$$F(z) = \frac{p+c}{z^c} \int_0^z t^{c-1} f(t)\, dt \tag{6.1}$$

also belongs to $S^*(p,n,\lambda,A,B)$.

Proof. Let

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \qquad (a_{k+p} \geq 0).$$

Then, from the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{k=1}^{\infty} \frac{p+c}{k+p+c} a_{k+p} z^{k+p}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{(k+p+n-1)(n+p+\lambda k)}{(n+p)!k!} \frac{p+c}{k+p+c} a_{k+p} \leq \frac{B-A}{1+B},$$

since $f(z) \in S^*(p,n,\lambda,A,B)$. Hence, by Theorem 1, $F(z) \in S^*(p,n,\lambda,A,B)$.

Theorem 8. Let c be a real number such that $c > -p$. If $F(z) \in S^*(p,n,\lambda,A,B)$, then the function $f(z)$ defined in (6.1) is p-valent for $|z| < R_p$, where

$$R_p = \inf_k \left[\frac{(k+p+n-1)!(n+p+\lambda k)(1+B)p(p+c)}{(n+p)!k!(B-A)(k+p)(k+p+c)} \right]^{\frac{1}{k}} \qquad (k \in N).$$

The result is sharp.

Proof. Let

$$F(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \qquad (a_{k+p} \geq 0).$$

It follows from (6.1)that

$$f(z) = \frac{z^{1-c}}{p+c} \frac{d}{dz}(z^c F(z)) = z^p - \sum_{k=1}^{\infty} \frac{k+p+c}{p+c} a_{k+p} z^{k+p}.$$

To prove the result, it suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| < p$$

for $|z| < R_p$.

The remaining part of the proof is similar to that of Theorem 5. The result is sharp for the function

$$f(z) = z^p - \frac{(n+p)!k!(B-A)(k+p+c)}{(k+p+n-1)!(n+p+\lambda k)(1+B)(p+c)} z^{k+p} \qquad (k \in N).$$

7. Factional integral

In 1978, Owa [5] gave the following definition for the fractional integral.

Definition 1. The fractional integral of order δ is defined by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\delta}} d\zeta,$$

where $\delta > 0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\delta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Theorem 9. Let a function

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \qquad (a_{k+p} \geq 0)$$

be in the class $S^*(p, n, \lambda, A, B)$. Then we have

$$|D_z^{-\delta} f(z)| \geq \frac{\Gamma(p+1)}{\Gamma(p+1+\delta)} |z|^{p+\delta} \left\{ 1 - \frac{(p+1)(B-A)}{(p+1+\delta)(n+p+\lambda)(1+B)} |z| \right\}$$

and

$$|D_z^{-\delta} f(z)| \leq \frac{\Gamma(p+1)}{\Gamma(p+1+\delta)} |z|^{p+\delta} \left\{ 1 + \frac{(p+1)(B-A)}{(p+1+\delta)(n+p+\lambda)(1+B)} |z| \right\}$$

for $0 < \delta < 1$ and $z \in U$. The result is sharp.

Proof. Let

$$\begin{aligned} F(z) &= \frac{\Gamma(p+1+\delta)}{\Gamma(p+1)} z^{-\delta} D_z^{-\delta} f(z) \\ &= z^p - \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1)\Gamma(p+1+\delta)}{\Gamma(k+p+1+\delta)\Gamma(p+1)} a_{k+p} z^{k+p} \\ &= z^p - \sum_{k=1}^{\infty} A(k) a_{k+p} z^{k+p}, \end{aligned}$$

where

$$A(k) = \frac{\Gamma(k+p+1)\Gamma(p+1+\delta)}{\Gamma(k+p+1+\delta)\Gamma(p+1)} \qquad (k \in N).$$

Since

$$0 < A(k) \leq A(1) = \frac{p+1}{p+1+\delta}$$

we have, with the help of Theorem 1,

$$\begin{aligned} |F(z)| &\geq |z|^p - A(1)|z|^{p+1} \sum_{k=1}^{\infty} a_{k+p} \\ &\geq |z|^p - \frac{(p+1)(B-A)}{(p+1+\delta)(n+p+\lambda)(1+B)} |z|^{p+1} \end{aligned}$$

and

$$\begin{aligned} |F(z)| &\leq |z|^p + A(1)|z|^{p+1} \sum_{k=1}^{\infty} a_{k+p} \\ &\leq |z|^p + \frac{(p+1)(B-A)}{(p+1+\delta)(n+p+\lambda)(1+B)} |z|^{p+1}, \end{aligned}$$

which prove the inequalities of Theorem 9. Further, the equalities are attained for the function given by (3.4).

Corollary 2. Under the hypothesis of Theorem 9, $D_z^{-\delta} f(z)$ is included in the disk with center at the origin and radius

$$\frac{\Gamma(p+1)}{\Gamma(p+1+\delta)} \left\{ 1 + \frac{(p+1)(B-A)}{(p+1+\delta)(n+p+\lambda)(1+B)} \right\}.$$

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