More on Shortest Confidence Intervals

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1. INTRODUCTION

Using a confidence interval to estimate an unknown parameter is a classical statis-tical technique that is introduced in almost every statistics course. To clarify the problem, we will follow Guenther (1969) and Ferentinos (1990) and consider the case where we are given a random sample $X_1, X_2, \ldots, X_n$ from a density $f(x, \theta)$ and a pivotal quantity $Q(X_1, X_2, \ldots, X_n, \theta)$ whose distribution dose not depend on θ. Finding a pivotal quantity will not be discussed, but the choice of a "good" pivotal quantity is essential for the resulting confidence interval to be useful. Usually the pivotal quantities are developed from either maximum likelihood estimates or sufficient statistics.

A confidence interval will be "constructed" in the following manner. Numbers a and b are chosen to satisfy the probability statement :

$$P[\, a < Q < b\,] = 1 - \alpha. \tag{1.1}$$

Often this expression can be solved for θ, in the form of an interval

$$P[\, W_1(X_1, X_2, \ldots, X_n) < \theta < W_2(X_1, X_2, \ldots, X_n)\,] = 1 - \alpha. \tag{1.2}$$

After observing data $x_1, x_2, \ldots, x_n$, the numbers $w_i = W_i(x_1, x_2, \ldots, x_n)$ are calculated and form the lower and upper endpoints of a $1-\alpha$ confidence interval based on Q.

Ferentinos (1990) has a discussion of finding pivotal quantities based on sufficient statis-tics in families with truncation parameters, and presents a different method for finding the shortest interval based on Q (Ferentinos 1990).

To illustrate the problem, consider two cases based on the normal distribution with expe-ction μ and variance σ^2, where a set of sufficient statistics are

$$\overline{X} = \sum_{k=1}^{n} X_k / n \quad \text{and} \quad S^2 = \sum_{k=1}^{n} (X_k - \overline{X})^2 / (n-1).$$

To find a confidence interval for μ , choose Q to be either $n^{1/2}(\overline{X}-\mu)/\sigma$ or $n^{1/2}(\overline{X}-\mu)/S$ depending on whether σ is assumed to be known or unknown. If σ is unknown then (1.2) becomes :

$$P[\overline{X} - Sb/n^{1/2} < \mu < \overline{X} - Sa/n^{1/2}] = 1-\alpha$$

This confidence interval has random length $S(b-a)/n^{1/2}$, and any realization will have length $s(b-a)/n^{1/2}$. In the following, we discuss the realized length of the confidence inte-rval.

If we seek a confidence interval for σ^2 (with μ unknown), choose Q to be $(n-1)S^2/\sigma^2$. The resulting confidence interval is implied by the statement

$$P[(n-1)S^2/b < \sigma^2 < (n-1)S^2/a] = 1-\alpha$$

and has realzied length $(n-1)s^2(1/a-1/b)$.

2. THE THEOREM

In both of the examples above, we see that the length of the confidence interval is proportional to the difference of a "simple" function evaluated at a and at b, typically $b-a$ for a location parameter and $1/a-1/b$ for a variance. Further, it can be seen that this difference is the integral from a to b of a "simple" function :

$$b - a = \int_a^b 1\,dt \quad \text{and} \tag{2.1}$$

$$1/a - 1/b = \int_a^b t^{-2}\,dt. \tag{2.2}$$

With this observation we can restate the problem of finding the shortest confidence interval as : find the values of a and b that minimize

$$\int_a^b g(x)\,dx$$

subject to

$$\int_a^b f(x)\,dx = 1-\alpha.$$

Or, more generally, find a region C to minimize

$$\int_C g(x)\,dx,$$

subject to

$$\int_C f(x)\,dx = 1-\alpha,$$

where f is a probability density and C is measaurable.

Theorem . Suppose $f(x)$ is a continuous density and $g(x)$ is continuous and positive, then to minimize

$$\int_C g(x)\,dx, \tag{2.3}$$

subject to

$$\int_C f(x)\,dx = 1-\alpha, \tag{2.4}$$

choose $C=\{x: f(x)/g(x) > \lambda\}$, where λ is chosen to satisfy (2.4).

Proof. Let R be any measurable set that satisfies (2.4) :

$$\begin{aligned} 0=\int_R f(t)\,dt-\int_C f(t)\,dt &= \int_{C'\cap R} f(t)\,dt-\int_{R'\cap C} f(t)\,dt \\ &\le \int_{C'\cap R}\lambda g(t)\,dt-\int_{R'\cap C}\lambda g(t)\,dt \\ &= \lambda\left\{\int_{C'\cap R} g(t)\,dt-\int_{R'\cap C} g(t)\,dt\right\} \\ &= \lambda\left\{\int_R g(t)\,dt-\int_C g(t)\,dt\right\} \end{aligned}$$

Since λ is positive, $\int_R g(t)\,dt \ge \int_C g(t)\,dt$, and C is the required minimizer.

3. EXAMPLES

Example 1 (Guenther 1969). Let $X_1, X_2, \ldots, X_n$ be a sample from a normal distribution with expection μ and variance σ^2 with σ unknown. A pivotal quantity Q based on the minimal sufficient statistics in $n^{1/2}(\overline{X}-\mu)/S$ which has a t distribu-tion with $n-1$ degrees of freedom. This t density is the function f of the theorem. As seen in the introduction, the length of this interval is proportional to $b-a$ so the func-tion g of the theorem is the constant function 1. The values of a and b which give the shortest confidence interval based on Q are given by the solution of the equa tion $f(a)=f(b)$ or $a=-b=-t_{n-1,\ \alpha/2}$ (Fig. 1).

Example 2 (Fenrentinos 1990). Let $X_1 \le X_2 \le \ldots \le x_n$ be the order statistics ass ociated with a random sample of size n from the density $1/2\theta$, $-\theta < x < \theta$. A pivotal quantity based on a minimal sufficient statistic is $Q = T/\theta$ where $T=\max(-X_1, X_n)$. Q has density function nq^{n-1} for $0<q<1$ and this is the function f of the theorem. The length of the confidence interval is $t(1/a - 1/b)$ so the function g of the theorem is $1/x^2$.

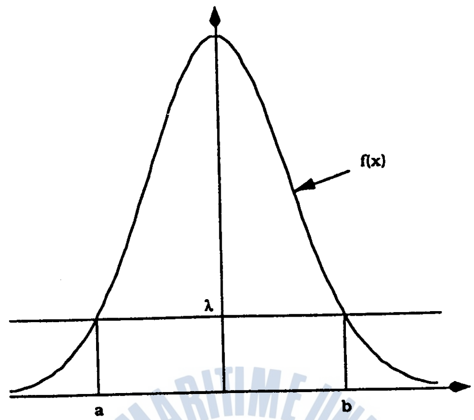

Figure 1. Confidence Interval for Normal Mean.

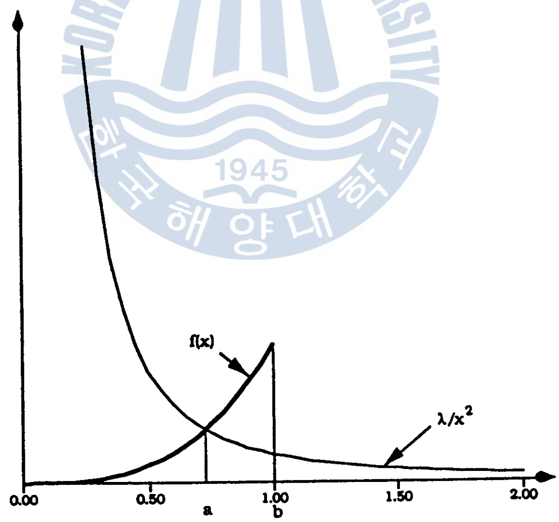

Figure 2. Confidence Interval for Truncation Parameter..

Using the theorem, we find that the confidence interval is given by the set of x's where $nx^{n-1} > \lambda / x^2$ (equivalently $nx^{n+1} > \lambda$) and $0 < x < 1$. Since x^{n+1} is increasing for x between 0 and 1, $b = 1$. To find a, observe that $P[a < Q < 1] = 1 - a^n = 1 - \alpha$ to obtain $a = \alpha^{1/n}$. Thus the shortest $1-\alpha$

confidence interval based on T is $(t,\ t\alpha^{-1/n})$ (Fig. 2).

Example 3. (Tate and Klett 1969). Let $X_1, X_2, \ldots, X_n$ be a sample from a normal (μ, σ^2) population with μ and σ^2 both unknown.. We want the shortest confidence interval for σ^2. A pivotal quantity based on sufficient statistics is $Q = (n-1)S^2/\sigma^2$,

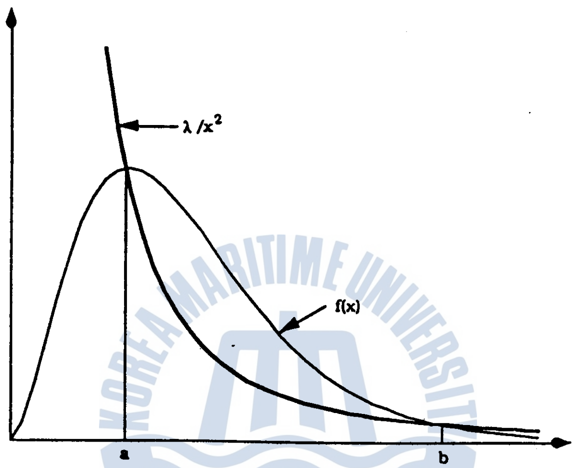

Figure 3. Confidence Interval for Normal Variance..

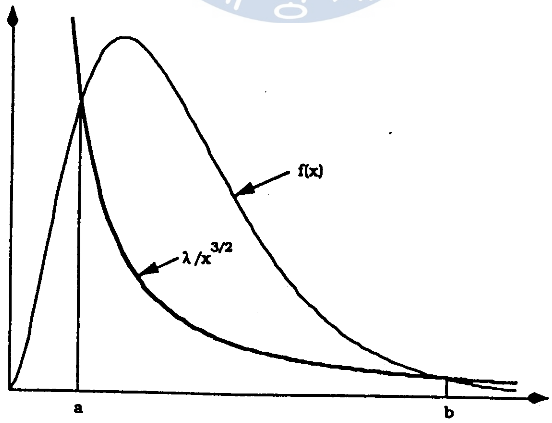

Figure 4. Confidence Interval for Normal Standard Deviation..

where $S^2 = \Sigma_{i=1}^{n}(X_i - \overline{X})^2/(n-1)$. Q has a chi-squared density $f_{n-1}(x)$. The con

-fidence interval is $(s^2/b,\ s^2/a)$ and the function g of the theorem is $1/x^2$. The va -lues of a and b are such that $f_{n-1}(a) = \lambda/a^2$ and $f_{n-1}(b) = \lambda/b^2$ and therefore determined by the equation $a^2 f_{n-1}(a) = b^2 f_{n-1}(b)$, together with the probability constraint (2.4). Tate and Klett (1969) have tables of the numerical solution of this equation for $n = 3\ (1)\ 30$ and $1-\alpha = .9,\ .95,\ .99,\ .995,\ .999$ (Fig. 3).

Example 4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal (μ, σ^2) popu -lation with μ and σ^2 both unknown. We want the shortest confidence interval for σ. Q and f are as in Example 3. The confidence interval is $(s/b^{1/2},\ s/a^{1/2})$ and the function g of the theorem is $1/x^{3/2}$. The values of a and b are determined by the equation $a^{3/2} f_{n-1}(a) = b^{3/2} f_{n-1}(b)$, together with the probability constraint (Fig. 4).

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