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$H^1(\Omega)$- NORM ERROR ANALYSIS OF THE HP-VERSION UNDER NUMERICAL QUADRATURE RULES

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ABSTRACT. we consider the hp-version to solve non-constant coefficients elliptic equations with Dirichlet boundary conditions on a bounded, convex polygonal domain Ω in R^2. In this paper we consider a family $G_p = \{I_m\}$ of numerical quadrature rules satisfying certain properties, which can be used for calculating the integrals. When the numerical quadrature rules $I_m \in G_p$ are used for calculating the integrals in the stiffness matrix of the variational form we will give its variational form and derive an error estimate of $\|u - \tilde{u}_p^h\|_{1,\Omega}$.

1. Introduction

The finite element method is a particular kind of Ritz-Galerkin procedure in which the approximating finite-dimensional subspaces are composed of piecewise polynomials defined on a partition of the given domain. The convergence is obtained by increasing the dimension of these subspaces in some manner. There are three versions of the finite element method. The h-version is the traditional approach obtained by fixing the degree p of the piecewise polynomials at some value (usually $p = 1, 2, 3$) and refining the mesh in order to achieve convergence. The p-version, in contrast, fixes the mesh and achieves the accuracy by increasing the degree p uniformly or selectively. The hp-version is the combination of both.

In this paper, to solve non-constant coefficients elliptic equations with Dirichlet boundary conditions on a bounded, convex polygonal domain Ω in R^2 we consider the hp-version with a quasi-uniform mesh and uniform p. In [6], I. Babuška and M. Suri already obtained the following optimal estimate for the hp-version:

(1.1) $\|u - u_p^h\|_{1,\Omega} \leq C p^{-(\sigma-1)} h^{min(p,\sigma-1)} \|u\|_{k,\Omega}$ for all $u \in H_0^\sigma(\Omega), \sigma \geq 1$,

where C is independent of u, h, and p [but depends on Ω and σ].

The above optimal result follows under the assumption that all integrations are performed exactly. In practice, to compute the integrals in the variational formulation of the discrete problem we need the numerical quadrature rule scheme. The integrals are seldom computed exactly. Thus we first consider a family $G_p = \{I_m\}$ of numerical quadrature rules satisfying certain properties, which can be used for calculating the integrals in the stiffness matrix of (2.17). Then, under the numerical quadrature rules we will give its variational form and derive an error estimate of $\|u - \tilde{u}_p^h\|_{1,\Omega}$ where $\tilde{u}_p^h$ is an approximation satisfying (3.6).

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2. Preliminaries

Let Ω be a bounded, convex polygonal domain in R^2 with the boundary Γ. Let $\mathcal{M} = \{\mathcal{J}^h\}, h \geq 0$ be a quasi-uniform, regular family of meshes $\mathcal{J}^h = \{\Omega_k^h\}$ defined on Ω, where Ω_k^h is a closed quadrilateral, and

$$(2.1) \qquad \max_{\Omega^h \in \mathcal{J}^h} \operatorname{diam}(\Omega^h) = h \quad \text{for all } \Omega^h, \mathcal{J}^h \in \mathcal{M}.$$

Further we assume that for each $\Omega_k^h \in \mathcal{J}^h$ there exists an invertible mapping T_k^h : $\widehat{\Omega} \to \Omega_k^h$ with the following correspondence:

$$(2.2) \qquad \widehat{x} \in \widehat{\Omega} \longleftrightarrow x = T_k^h(\widehat{x}) \in \Omega_k^h,$$

and

$$(2.3) \qquad \widehat{t} \in U_p(\widehat{\Omega}) \longleftrightarrow t = \widehat{t} \circ \left(T_k^h\right)^{-1} \in U_p(\Omega_k^h),$$

where $\widehat{\Omega}$ denotes the reference elements $\widehat{I}^2 = [-1,1]^2$ in R^2 ,

$$(2.4) \qquad U_p(\widehat{\Omega}) = \{\, \widehat{t} : \widehat{t} \text{ is a polynomial of degree} \leq p \text{ in each variable on } \widehat{\Omega}\,\},$$

and

$$(2.5) \qquad U_p(\Omega_k^h) = \{\, t : \widehat{t} = t \circ T_k^h \in U_p(\widehat{\Omega})\,\}.$$

We now consider the following model problem of elliptic equations :

Find $u \in H_0^1(\Omega)$, such that

$$(2.6) \qquad -\operatorname{div}(a\nabla u) = f \quad \text{in} \quad \Omega \subset R^2,$$

where two functions a and f satisfy a compatibility condition to ensure a solution exists, and

$$(2.7) \qquad H_0^1(\Omega) = \{u \in H^1(\Omega) : u \text{ vanishes on } \Gamma\}.$$

For the sake of simplicity, we assume that

$$(2.8) \qquad 0 < A_1 \leq a(x) \leq A_2 \quad \text{for all} \quad x \in \Omega, \text{and}$$

$$(2.9) \qquad f \in L_2(\Omega).$$

In addition, we also assume that there exists a constant $M \geq 1$ such that

$$(2.10) \quad \|T_k^h\|_{m,\infty,\widehat{\Omega}}\,, \quad \|\left(T_k^h\right)^{-1}\|_{m,\infty,\Omega_k^h} \leq A \quad \text{for} \quad 0 \leq m \leq M,$$

$$(2.11) \quad \|\widehat{J}_k^h\|_{m,\infty,\widehat{\Omega}}\,, \quad \|\left(\widehat{J}_k^h\right)^{-1}\|_{m,\infty,\Omega_k^h} \leq A \quad \text{for} \quad 0 \leq m \leq M-1,$$

where $\widehat{J}_k^h$ and $\left(\widehat{J}_k^h\right)^{-1}$ denote the Jacobians of T_k^h and $\left(T_k^h\right)^{-1}$ respectively. Then, as seen in [8,theorem 3.1.2], we obtain the following correspondence: For any $\alpha \in [1,\infty]$, $0 \leq m \leq M$,

$$(2.12) \qquad \widehat{t} \in W^{m,\alpha}(\widehat{\Omega}) \longleftrightarrow t = \widehat{t} \circ \left(T_k^h\right)^{-1} \in W^{m,\alpha}(\Omega_k^h)$$

with norm equivalence

$$(2.13) \qquad C_1 h^{(m-\frac{2}{\alpha})}\|t\|_{m,\alpha,\Omega_k^h} \leq \|\widehat{t}\|_{m,\alpha,\widehat{\Omega}} \leq C_2 h^{(m-\frac{2}{\alpha})}\|t\|_{m,\alpha,\Omega_k^h},$$

with the subscript α omitted when $\alpha = 2$. Namely, we have

$$(2.14) \qquad C_1 h^{(m-1)}\|t\|_{m,\Omega_k^h} \leq \|\widehat{t}\|_{m,\widehat{\Omega}} \leq C_2 h^{(m-1)}\|t\|_{m,\Omega_k^h}.$$

Let us define

(2.15) $S_p^h(\Omega) = \{u \in H^1(\Omega) \,:\, u_{\Omega_k^h} \circ (T_k^h) \in U_p(\widehat{\Omega}) \text{ for all } \Omega_k^h \in \mathcal{J}^h\}$,

where $u_{\Omega_k^h}$ denotes the restriction of $u \in H^1(\Omega)$ to $\Omega_k^h \in \mathcal{J}^h$, and

$$S_{p,0}^h(\Omega) = S_p^h(\Omega) \cap H_0^1(\Omega). \tag{2.16}$$

Then, using the hp−version of the finite element method with the mesh $\mathcal{J}^h = \{\Omega_k^h\}$ we obtain the following discrete variational form of (2.6):

Find $u_p^h \in S_{p,0}^h(\Omega)$ satisfying

$$B(u_p^h, v_p^h) = (f, v_p^h)_\Omega \quad \text{for all} \quad v_p^h \in S_{p,0}^h(\Omega), \tag{2.17}$$

where

$$B(u,v) = \int_\Omega a\nabla u \cdot \nabla v \, dx, \tag{2.18}$$

the usual inner product

$$(f,v)_\Omega = \int_\Omega fv \, dx. \tag{2.19}$$

Let us now give some approximation results which will be used later.

LEMMA 2.1. *For each integer $l \ge 0$, there exists a sequence of projections* $\Pi_p^l \,:\, H^l(\widehat{\Omega}) \to U_p(\widehat{\Omega}), \quad p = 1,2,3,\cdots$ *such that*

(2.20) $\Pi_p^l \widehat{v}_p = \widehat{v}_p$ *for all* $\widehat{v}_p \in U_p(\widehat{\Omega})$,

(2.21) $\|\widehat{u} - \Pi_p^l \widehat{u}\|_{s,\widehat{\Omega}} \le C p^{-(r-s)} \|\widehat{u}\|_{r,\widehat{\Omega}}$ *for all* $\widehat{u} \in H^r(\widehat{\Omega})$ *with* $0 \le s \le l \le r$.

Proof. See [9, *Lemma* 3.1]. □

LEMMA 2.2 *Let $\widehat{u} \in H^r(\widehat{\Omega})$ with $r \ge 2$. Then the projection Π_p^2 from lemma 1.1 satisfies*

(2.22) $\|\widehat{u} - \Pi_p^2 \widehat{u}\|_{0,\infty,\widehat{\Omega}} \le C p^{-(r-1)} \|\widehat{u}\|_{r,\widehat{\Omega}}$.

Proof. By interpolation results (see [9, theorem 3.2] and [7, theorem 6.2.4]) we have that for $0 < \varepsilon \le \frac{1}{2}$,

$$\|\widehat{u} - \Pi_p^2 \widehat{u}\|_{0,\infty,\widehat{\Omega}} \le C \|\widehat{u} - \Pi_p^2 \widehat{u}\|_{1+\varepsilon,\widehat{\Omega}}^{\frac{1}{2}} \|\widehat{u} - \Pi_p^2 \widehat{u}\|_{1-\varepsilon,\widehat{\Omega}}^{\frac{1}{2}}. \tag{2.23}$$

We also have from lemma 1.1 that

$$\|\widehat{u} - \Pi_p^2 \widehat{u}\|_{r,\widehat{\Omega}} \le C p^{-(s-r)} \|\widehat{u}\|_{s,\widehat{\Omega}} \text{ for } 0 \le r \le 2 \le s. \tag{2.24}$$

Hence, taking $r = 1 + \varepsilon$ and $r = 1 - \varepsilon$ in (1.24) we obtain

$$\|\widehat{u} - \Pi_p^2 \widehat{u}\|_{1+\varepsilon,\widehat{\Omega}}^{\frac{1}{2}} \|\widehat{u} - \Pi_p^2 \widehat{u}\|_{1-\varepsilon,\widehat{\Omega}}^{\frac{1}{2}} \le C p^{-(s-1)} \|\widehat{u}\|_{s,\widehat{\Omega}}, \tag{2.25}$$

which completes the proof from (2.23). □

LEMMA 2.3. *Suppose that $T_k^h : \widehat{\Omega} \longrightarrow \Omega_k^h$ is an invertible affine mapping. Then for any $u \in H^\sigma(\Omega), \sigma \geq 0$ we have*

$$\inf_{\widehat{v} \in U_p(\widehat{\Omega})} \|\widehat{u}_{\Omega_k^h} - \widehat{v}\|_{\sigma,\widehat{\Omega}} \leq Ch^\mu \|u_{\Omega_k^h}\|_{\sigma,\Omega_k^h}, \tag{2.26}$$

where $\mu = \min(p, \sigma - 1)$ and C is independent of h, p and u.

Proof. The proof is given in [6]. □

LEMMA 2.4. *For each $u \in H^\sigma(\Omega)$ and $\Omega_k^h \in \mathcal{J}^h$ there exists a sequence $z_p^h \in U_p(\Omega_k^h), p = 1, 2, \cdots$ such that for any $0 \leq r \leq \sigma$*

$$\|u_{\Omega_k^h} - z_p^h\|_{r,\Omega_k^h} \leq Ch^{(\mu - r + 1)} p^{-(\sigma - r)} \|u_{\Omega_k^h}\|_{\sigma,\Omega_k^h} \text{ for all } \Omega_k^h \in \mathcal{J}^h, \tag{2.27}$$

where $\mu = \min(p, \sigma - 1)$ and C is independent of h, p and u.

Proof. See [6, Lemma 4.5]. □

3. The hp-version under numerical quadrature rules

We consider numerical quadrature rules I_m defined on the reference element $\widehat{\Omega}$ by

$$I_m(\widehat{f}) = \sum_{i=1}^{n(m)} \widehat{\omega}_i^m \widehat{f}(\widehat{x}_i^m) \sim \int_{\widehat{\Omega}} \widehat{f}(\widehat{x})\, d\widehat{x}, \tag{3.1}$$

where m is a positive integer. Let $G_p = \{I_m\}$ be a family of quadrature rules I_m with respect to $U_p(\widehat{\Omega})$, $p = 1, 2, 3, \cdots$, satisfying the following properties: For each $I_m \in G_p$,

(K1) $\widehat{\omega}_i^m > 0$ and $\widehat{x}_i^m \in \widehat{\Omega}$ for $i = 1, \cdots, n(m)$.

(K2) $I_m(\widehat{f}^2) \leq C_1 \|\widehat{f}\|_{0,\widehat{\Omega}}^2$ for all $\widehat{f} \in U_p(\widehat{\Omega})$.

(K3) $C_2 \|\widetilde{f}\|_{0,\widehat{\Omega}}^2 \leq I_m(\widetilde{f}^2)$ for all $\widetilde{f} \in \widetilde{U}_p(\widehat{\Omega})$,

where $\widetilde{U}_p(\widehat{\Omega}) = \{ \frac{\partial \widehat{f}}{\partial \widehat{x}_i} : \widehat{f} \in U_p(\widehat{\Omega}) \} \subset U_p(\widehat{\Omega})$

(K4) $I_m(\widehat{f}) = \int_{\widehat{\Omega}} \widehat{f}(\widehat{x})\, d\widehat{x}$ for all $\widehat{f} \in U_{d(m)}(\widehat{\Omega})$,

where $d(m) \geq \widetilde{d}(p) > 0$.

We also get a family $G_{p,\Omega} = \{I_{m,\Omega}\}$ of numerical quadrature rules with respect to $S_p^h(\Omega)$ defined by

$$(3.2)\quad I_{m,\Omega_k^h}(f_{\Omega_k^h}) = \sum_{j=1}^{n(m)} w_j^k f_{\Omega_k^h}(x_j^m) = \sum_{j=1}^{n(m)} \widehat{w}_j^m \widehat{J}_k^h(\widehat{x}_j^m)(f_{\Omega_k^h}\circ T_k^h)(\widehat{x}_j^m) = I_m(\widehat{J}_k^h \widehat{f}_{\Omega_k^h})$$

and

$$(3.3)\qquad I_{m,\Omega}(f) = \sum_{\Omega_k^h\in\mathcal{J}^h} I_{m,\Omega_k^h}(f_{\Omega_k^h}).$$

Here, one may employ numerical quadrature rules schemes for computing the integrals in the discrete variational form (1.17). Especially, since the model problem (1.6) is a non-constant coefficients elliptic problem the numerical quadrature rules $I_m \in G_p$ can be used for calculating the integrals in the stiffness matrix. Thus, we denote by DF the 2×2 Jacobian matrix of $F: R^2 \to R^2$, and define two discrete inner products

$$(3.4)\qquad (u,v)_{m,\Omega_k^h} = I_{m,\Omega_k^h}\big((uv)_{\Omega_k^h}\big) = I_m\big(\widehat{J_k^h}\,\widehat{(uv)_{\Omega_k^h}}\big) \text{ on } \Omega_k^h \in \mathcal{J}^h,$$

$$(3.5)\qquad (u,v)_{m,\Omega} = \sum_{\Omega_k^h\in\mathcal{J}^h} (u,v)_{m,\Omega_k^h} \text{ on } \Omega.$$

Then, under the assumption that all integrations in the load vector of (1.17) are performed exactly, using the quadrature rules $I_m \in G_p$ for computing the integrals in the stiffness matrix of (1.17) we obtain the following actual problem of (1.17): Find $\widetilde{u}_p^h \in S_{p,0}^h(\Omega)$, such that

$$(3.6)\qquad B_{m,\Omega}(\widetilde{u}_p^h, v_p^h) = (f, v_p^h)_\Omega \text{ for all } v_p^h \in S_{p,0}^h(\Omega),$$

where

$$B_{m,\Omega}(\widetilde{u}_p^h, v_p^h) = I_{m,\Omega}(a\nabla\widetilde{u}_p^h\cdot\nabla v_p^h)$$
$$= \sum_{\Omega_k^h\in\mathcal{J}^h} I_{m,\Omega_k^h}\big(a_{\Omega_k^h}\nabla(\widetilde{u}_p^h)_{\Omega_k^h}\cdot\nabla(v_p^h)_{\Omega_k^h}\big)$$
$$= \sum_{\Omega_k^h\in\mathcal{J}^h} I_m\Big(\widehat{J_k^h}\widehat{a_{\Omega_k^h}}\Big[\big(\widehat{DT_k^{h^{-1}}}\big)^t\big(\nabla\widehat{(\widetilde{u}_p^h)_{\Omega_k^h}}\big)\Big]^t\Big[\big(\widehat{DT_k^{h^{-1}}}\big)^t\big(\nabla\widehat{(v_p^h)_{\Omega_k^h}}\big)\Big]\Big).$$

Here, if we let $\big(DT_k^{h^{-1}}\big)\big(DT_k^{h^{-1}}\big)^t = \begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$, then $\widehat{(a_{ij})_{\Omega_k^h}} = \widehat{J_k^h}\widehat{(b_{ij})_{\Omega_k^h}}$ are the entries of the matrix $\widehat{J_k^h}\big(\widehat{DT_k^{h^{-1}}}\big)\big(\widehat{DT_k^{h^{-1}}}\big)^t$. For the simplicity of notation, if the restrictions $\widehat{a_{\Omega_k^h}}$, $\widehat{(a_{ij})_{\Omega_k^h}}$, $\widehat{(\widetilde{u}_p^h)_{\Omega_k^h}}$ and $\widehat{(v_p^h)_{\Omega_k^h}}$ are simply denoted by $\widehat{a}$, $\widehat{a_{ij}}$, $\widehat{\widetilde{u}_p^h}$ and $\widehat{v_p^h}$ respectively, then we have

$$(3.7)\quad B_{m,\Omega}(\widetilde{u}_p^h, v_p^h)$$
$$= \sum_{\Omega_k^h\in\mathcal{J}^h} I_m\Big(\widehat{J_k^h}\widehat{a_{\Omega_k^h}}\big(\nabla\widehat{(\widetilde{u}_p^h)_{\Omega_k^h}}\big)^t\big(\widehat{DT_k^{h^{-1}}}\big)\big(\widehat{DT_k^{h^{-1}}}\big)^t\big(\nabla\widehat{(v_p^h)_{\Omega_k^h}}\big)\Big)$$
$$= \sum_{\Omega_k^h\in\mathcal{J}^h} I_m\left(\widehat{a}\begin{pmatrix}\frac{\partial\widehat{\widetilde{u}_p^h}}{\partial\widehat{x}_1}\\ \frac{\partial\widehat{\widetilde{u}_p^h}}{\partial\widehat{x}_2}\end{pmatrix}^t\begin{pmatrix}\widehat{a_{11}} & \widehat{a_{12}}\\ \widehat{a_{21}} & \widehat{a_{22}}\end{pmatrix}\begin{pmatrix}\frac{\partial\widehat{v_p^h}}{\partial\widehat{x}_1}\\ \frac{\partial\widehat{v_p^h}}{\partial\widehat{x}_2}\end{pmatrix}\right)$$
$$= \sum_{\Omega_k^h\in\mathcal{J}^h}\sum_{i,j=1}^{2}\Big(\widehat{a}\widehat{a_{ij}}\frac{\partial\widehat{\widetilde{u}_p^h}}{\partial\widehat{x}_i}, \frac{\partial\widehat{v_p^h}}{\partial\widehat{x}_j}\Big)_{m,\widehat{\Omega}}$$

$$= \sum_{\Omega^h_k \in \mathcal{J}^h} \sum_{i,j=1}^{2} (\widehat{a}\widehat{a_{ij}} \frac{\partial \widehat{\widetilde{u}^h_p}}{\partial \widehat{x}_i}, \frac{\partial \widehat{v^h_p}}{\partial \widehat{x}_j})_{m,\widehat{\Omega}}.$$

Let us now derive an estimate of the error $\|u - \widetilde{u}^h_p\|_{1,\Omega}$ for the $hp-$ version under numerical quadrature rules I_m. In fact, $\|u - \widetilde{u}^h_p\|_{1,\Omega}$ depends on two separate terms. The first dependence is on the error $\|u - u^h_p\|_{1,\Omega}$ given in (1.1). Next, the error will depend upon the smoothness of a. We will start with the following Lemma.

LEMMA 3.1. *Let u be the exact solution of (2.6) and u^h_p that of (2.17). Let $\widetilde{u}^h_p$ be an approximate solution of u which satisfies a discrete variational form (3.6). Then there exists a constant C independent of m such that*

(3.8) $\|u - \widetilde{u}^h_p\|_{1,\Omega}$

$$\leq C \inf_{v^h_p \in S^h_{p,0}(\Omega)} \{\|u - v^h_p\|_{1,\Omega} + \sup_{w^h_p \in S^h_{p,0}(\Omega)} \frac{|B(u^h_p, w^h_p) - B_{m,\Omega}(v^h_p, w^h_p)|}{\|w^h_p\|_{1,\Omega}}\}.$$

Proof. Let v^h_p be an arbitrary element in $S^h_{p,0}(\Omega)$. Then we have

(3.9) $\quad \|u - \widetilde{u}^h_p\|_{1,\Omega} \leq \|u - v^h_p\|_{1,\Omega} + \|v^h_p - \widetilde{u}^h_p\|_{1,\Omega}.$

From the ellipticity of $B_{m,\Omega}(\cdot,\cdot)$, for a constant $C_1 > 0$

$$\begin{aligned} (3.10) \quad C_1\|v^h_p - \widetilde{u}^h_p\|^2_{1,\Omega} &\leq B_{m,\Omega}(v^h_p - \widetilde{u}^h_p, v^h_p - \widetilde{u}^h_p) \\ &= |B_{m,\Omega}(v^h_p, v^h_p - \widetilde{u}^h_p) - (f, v^h_p - \widetilde{u}^h_p)| \\ &= |B_{m,\Omega}(v^h_p, v^h_p - \widetilde{u}^h_p) - B(u^h_p, v^h_p - \widetilde{u}^h_p)|. \end{aligned}$$

Hence, taking the infimum with respect to $v^h_p \in S^h_{p,0}(\Omega)$ we have

(3.11) $\|u - \widetilde{u}^h_p\|_{1,\Omega}$

$$\leq C \inf_{v^h_p \in S^h_{p,0}(\Omega)} \{\|u - v^h_p\|_{1,\Omega} + \frac{|B(u^h_p, v^h_p - \widetilde{u}^h_p) - B_{m,\Omega}(v^h_p, v^h_p - \widetilde{u}^h_p)|}{\|v^h_p - \widetilde{u}^h_p\|_{1,\Omega}}\}.$$

The Lemma follows from taking $w^h_p = v^h_p - \widetilde{u}^h_p \in S^h_{p,0}(\Omega)$. □

LEMMA 3.2. *Let $\widehat{u_p}, \widehat{w_p} \in U_p(\widehat{\Omega})$ and $\widehat{g} \in L_\infty(\widehat{\Omega})$. Then, for all $\widehat{v^1_q}, \widehat{v^2_q} \in U_q(\widehat{\Omega})$, $\widehat{g_r} \in U_r(\widehat{\Omega})$ with $0 < q \leq p$ and $r = d(m) - p - q > 0$ we have*

$$\begin{aligned} (3.12) \quad & |(\widehat{g}\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g}\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}| \\ &\leq C\{\|\widehat{g_r}\|_{0,\infty,\widehat{\Omega}}\|\widehat{u_p} - \widehat{v^1_q}\|_{0,\widehat{\Omega}}\|\widehat{u_p} - \widehat{v^2_q}\|_{0,\widehat{\Omega}} \\ &\quad + \|\widehat{g} - \widehat{g_r}\|_{0,\infty,\widehat{\Omega}}\|\widehat{u_p}\|_{0,\widehat{\Omega}}\ \|\widehat{u_p}\|_{0,\widehat{\Omega}}\}, \end{aligned}$$

where C is independent of p, q and m.

Proof. For any $\widehat{g_r} \in U_r(\widehat{\Omega})$ we have

$$(3.13)\quad |(\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}| \\ \le |(\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}}| + |(\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}| \\ + |(\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}} - (\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}|.$$

Thank to (K4),

$$(3.14)\quad (\widehat{g_r}\widehat{v_q^1}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{v_q^1}, \widehat{u_p})_{m,\widehat{\Omega}} = 0 \quad \text{for any} \quad \widehat{v_q^1} \in U_q(\widehat{\Omega}), \text{ and} \\ (\widehat{g_r}\widehat{u_p}, \widehat{v_q^2})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{v_q^2})_{m,\widehat{\Omega}} = 0 \quad \text{for any} \quad \widehat{v_q^2} \in U_q(\widehat{\Omega}).$$

Hence,

$$(3.15)\quad |(\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}| \\ \le |(\widehat{g_r}\widehat{u_p}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}}| \\ + |(\widehat{g_r}\widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}}|.$$

By the Schwarz inequality we obtain

$$(3.16)\quad |(\widehat{g_r}\widehat{u_p}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}}| \\ \le (\widehat{g_r}(\widehat{u_p} - \widehat{v_q^1}), \widehat{g_r}(\widehat{u_p} - \widehat{v_q^1}))_{\widehat{\Omega}}^{\frac{1}{2}} (\widehat{u_p} - \widehat{v_q^2}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}}^{\frac{1}{2}} \\ \le C\,\|\widehat{g_r}\|_{0,\infty,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^1}\|_{0,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^2}\|_{0,\widehat{\Omega}}.$$

Also, from (K2) we have

$$(3.17)\quad |(\widehat{g_r}\widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}}| \\ \le (\widehat{g_r}(\widehat{u_p} - \widehat{v_q^1}), \widehat{g_r}(\widehat{u_p} - \widehat{v_q^1}))_{m,\widehat{\Omega}}^{\frac{1}{2}} (\widehat{u_p} - \widehat{v_q^2}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}}^{\frac{1}{2}} \\ \le C\|\widehat{g_r}\|_{0,\infty,\widehat{\Omega}} (\widehat{u_p} - \widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^1})_{m,\widehat{\Omega}}^{\frac{1}{2}} (\widehat{u_p} - \widehat{v_q^2}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}}^{\frac{1}{2}} \\ \le C\|\widehat{g_r}\|_{0,\infty,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^1}\|_{0,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^2}\|_{0,\widehat{\Omega}}.$$

Hence, combining (3.16) and (3.17) we estimate

$$(3.18)\quad |(\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}| \\ \le C\,\|\widehat{g_r}\|_{0,\infty,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^1}\|_{0,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^2}\|_{0,\widehat{\Omega}}.$$

Similarly, since $\widehat{g} \in L_\infty(\widehat{\Omega})$ we obtain

$$(3.19)\quad |(\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}}| \\ \le ((\widehat{g} - \widehat{g_r})\widehat{u_p}, (\widehat{g} - \widehat{g_r})\widehat{u_p})_{\widehat{\Omega}}^{\frac{1}{2}} (\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}}^{\frac{1}{2}} \\ \le C\,\|\widehat{g} - \widehat{g_r}\|_{0,\infty,\widehat{\Omega}} \|\widehat{u_p}\|_{0,\widehat{\Omega}} \|\widehat{u_p}\|_{0,\widehat{\Omega}},$$

and

$$(3.20)\quad |(\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}} - (\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}|$$

$$
\begin{aligned}
&\le ((\widehat{g}_r - f)\widehat{u_p}, (\widehat{g}_r - \widehat{g})\widehat{u_p})^{\frac{1}{2}}_{m,\widehat{\Omega}} (\widehat{u_p}, \widehat{u_p})^{\frac{1}{2}}_{m,\widehat{\Omega}} \\
&\le C\, \|\widehat{g}_r - \widehat{g}\|_{0,\infty,\widehat{\Omega}} (\widehat{u_p}, \widehat{u_p})^{\frac{1}{2}}_{m,\widehat{\Omega}} (\widehat{u_p}, \widehat{u_p})^{\frac{1}{2}}_{m,\widehat{\Omega}} \\
&\le C\, \|\widehat{g}_r - \widehat{g}\|_{0,\infty,\widehat{\Omega}} \|\widehat{u_p}\|_{0,\widehat{\Omega}} \|\widehat{u_p}\|_{0,\widehat{\Omega}} \,.
\end{aligned}
$$

The lemma follows from (3.18), (3.19), (3.20) and (3.13). □

As seen in Lemma 3.1, the last dependence of $\|u - \widetilde{u}_p^h\|_{1,\Omega}$ is on the smoothness of a. In this connection, we let

$$
(3.21) \qquad M_{p,q} = \max_{\Omega_k^h \in \mathcal{J}^h} \max_{i,j} \|\widehat{a}\widehat{a_{ij}}\|_{p,q,\widehat{\Omega}},
$$

where the subscript q will be omitted when $q = 2$. Then, we obtain the following results which give an estimate for the last term of the right side in (3.8).

LEMMA 3.3. *Let* $I_m \in G_p$ *be a quadrature rule defined on* $\widehat{\Omega} \subset R^2$, *which satisfies* $d(m) - p - 1 > 0$. *Let* $u \in H^\sigma(\Omega)$, $a \in H^\alpha(\Omega)$ *and* $\widehat{a}_{ij} \in H^\rho(\widehat{\Omega})$ *for* $i, j = 1, 2$, *such that* $\lambda = \min(\alpha, \rho) \ge 2$. *Then, for any* $w_p^h \in S_{p,0}^h(\Omega)$ *and an approximation* u_p^h *which satisfies (2.17) we have*

$$
(3.22) \qquad \frac{|\, B(u_p^h, w_p^h) - B_{m,\Omega}(u_p^h, w_p^h)\,|}{\|w_p^h\|_{1,\Omega}}
$$
$$
\le C\{(r^{-(\lambda-1)} M_\lambda + M_{0,\infty})(\|u - u_p^h\|_{1,\Omega} + q^{-(\sigma-1)} h^{(\sigma-1)} \|u\|_{\sigma,\Omega}) + r^{-(\lambda-1)} M_\lambda \|u\|_{1,\Omega}\},
$$

where q *is a positive integer such that* $0 < q \le p$ *and* $r = d(m) - p - q > 0$.

Proof. For arbitrary $w_p^h \in S_{p,0}^h(\Omega)$ we have

$$
(3.23) \qquad |\, B(u_p^h, w_p^h) - B_{m,\Omega}(u_p^h, w_p^h)\,|
$$
$$
\le C \max_{\Omega_k^h \in \mathcal{J}^h} \max_{i,j} \left| \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p^h}{\partial \widehat{x}_j} \right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p^h}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} \right|.
$$

For any $\widehat{a}_{ij}$ $i, j = 1, 2$ and $\Omega_k^h \in \mathcal{J}^h$ we let q be any integer such that $0 < q \le p$ and $r = d(m) - p - q > 0$. Then since $\widehat{a}\,\widehat{a}_{ij} \in L_\infty(\widehat{\Omega})$, due to Lemma 3.2 with $\widehat{v}_q^1 = \dfrac{\partial}{\partial \widehat{x}_i}(\Pi_q^1 \widehat{u}_p^h)$ $\widehat{v}_q^2 = \dfrac{\partial \Pi_q^1 \widehat{w_p^h}}{\partial \widehat{x}_j} \in U_q(\widehat{\Omega})$ and $\widehat{g}_r = \Pi_r^2(\widehat{a}\,\widehat{a}_{ij})$, we have

$$
(3.24) \qquad \left| \left(\widehat{a}\,\widehat{a_{ij}} \frac{\partial \widehat{u_p^h}}{\partial \widehat{x}_i}, \frac{\partial \widehat{w_p^h}}{\partial \widehat{x}_j} \right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a_{ij}} \frac{\partial \widehat{u_p^h}}{\partial \widehat{x}_i}, \frac{\partial \widehat{w_p^h}}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} \right|
$$
$$
\le C\{\, \|\Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}} \left\| \frac{\partial \widehat{u_p^h}}{\partial \widehat{x}_i} - \frac{\partial \Pi_q^1 \widehat{u}}{\partial \widehat{x}_i} \right\|_{0,\widehat{\Omega}} \left\| \frac{\partial \widehat{w_p^h}}{\partial \widehat{x}_j} - \frac{\partial \Pi_q^1 \widehat{w_p^h}}{\partial \widehat{x}_j} \right\|_{0,\widehat{\Omega}}
$$

$$+ \|\widehat{a}\,\widehat{a_{ij}} - \Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}} \|\frac{\partial \widehat{u_p^h}}{\partial \widehat{x}_i}\|_{0,\widehat{\Omega}} \|\frac{\partial \widehat{w_p^h}}{\partial \widehat{x}_j}\|_{0,\widehat{\Omega}} \}.$$

Since $\widehat{a}\,\widehat{a_{ij}} \in H^\lambda(\widehat{\Omega})$ with $\lambda = \min(\alpha,\rho) \geq 2$ we obtain from lemma 2.2 and (2.14) that

$$(3.25) \qquad \|\widehat{a}\,\widehat{a_{ij}} - \Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}} \|\frac{\partial \widehat{u_p^h}}{\partial \widehat{x}_i}\|_{0,\widehat{\Omega}} \|\frac{\partial \widehat{w_p^h}}{\partial \widehat{x}_j}\|_{0,\widehat{\Omega}}$$
$$\leq C r^{-(\lambda-1)} \|\widehat{a}\,\widehat{a_{ij}}\|_{\lambda,\widehat{\Omega}} (\|\widehat{u} - \widehat{u_p^h}\|_{1,\widehat{\Omega}} + \|\widehat{u}\|_{1,\widehat{\Omega}}) \|\widehat{w_p^h}\|_{1,\widehat{\Omega}}$$
$$\leq C r^{-(\lambda-1)} M_\lambda (\|u - u_p^h\|_{1,\Omega_k^h} + \|u\|_{1,\Omega_k^h}) \|w_p^h\|_{1,\Omega_k^h}.$$

Further, it follows from lemma 2.1, lemma 2.2 and (2.14) that

$$(3.26) \quad \|\Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}} \|\frac{\partial \widehat{u_p^h}}{\partial \widehat{x}_i} - \frac{\partial \Pi_q^1 \widehat{u}}{\partial \widehat{x}_i}\|_{0,\widehat{\Omega}} \|\frac{\partial \widehat{w_p^h}}{\partial \widehat{x}_j} - \frac{\partial \Pi_q^1 \widehat{w_p^h}}{\partial \widehat{x}_j}\|_{0,\widehat{\Omega}}$$
$$\leq C \{ \|\widehat{a}\,\widehat{a_{ij}} - \Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}}$$
$$+ \|\widehat{a}\,\widehat{a_{ij}}\|_{0,\infty,\widehat{\Omega}} \} \|\widehat{u_p^h} - \Pi_q^1 \widehat{u}\|_{1,\widehat{\Omega}} \|\widehat{w_p^h} - \Pi_q^1 \widehat{w_p^h}\|_{1,\widehat{\Omega}}$$
$$\leq C \{ \|\widehat{a}\,\widehat{a_{ij}} - \Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}} + M_{0,\infty} \} \{ \|\widehat{u} - \widehat{u_p^h}\|_{1,\widehat{\Omega}}$$
$$+ \|\widehat{u} - \Pi_q^1 \widehat{u}\|_{1,\widehat{\Omega}} \} \|\widehat{w_p^h} - \Pi_q^1 \widehat{w_p^h}\|_{1,\widehat{\Omega}}$$
$$\leq C \{ r^{-(\lambda-1)} \|\widehat{a}\,\widehat{a_{ij}}\|_{\lambda,\widehat{\Omega}} + M_{0,\infty} \} \{ \|\widehat{u} - \widehat{u_p^h}\|_{1,\widehat{\Omega}}$$
$$+ q^{-(\sigma-1)} \|\widehat{u}\|_{\sigma,\widehat{\Omega}} \} \|\widehat{w_p^h}\|_{1,\widehat{\Omega}}$$
$$\leq C \{ r^{-(\lambda-1)} M_\lambda + M_{0,\infty} \} \{ \|u - u_p^h\|_{1,\Omega_k^h}$$
$$+ q^{-(\sigma-1)} h^{(\sigma-1)} \|u\|_{\sigma,\Omega_k^h} \} \|w_p^h\|_{1,\Omega_k^h},$$

where C is independent of p and q.

Thus, substituting (3.25) and (3.26) in (3.24) we have

$$(3.27) \qquad | B(u_p^h, w_p^h) - B_{m,\Omega}(u_p^h, w_p^h) |$$
$$\leq C \sum_{\Omega_k^h \in \mathcal{J}^h} \max_{i,j} \left| \left(\widehat{a}\,\widehat{a_{ij}} \frac{\partial \widehat{u_p^h}}{\partial \widehat{x}_i}, \frac{\partial \widehat{w_p^h}}{\partial \widehat{x}_j} \right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a_{ij}} \frac{\partial \widehat{u_p^h}}{\partial \widehat{x}_i}, \frac{\partial \widehat{w_p^h}}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} \right|$$
$$\leq C \sum_{\Omega_k^h \in \mathcal{J}^h} \{ (r^{-(\lambda-1)} M_\lambda + M_{0,\infty}) (\|u - u_p^h\|_{1,\Omega_k^h}$$
$$+ q^{-(\sigma-1)} h^{(\sigma-1)} \|u\|_{\sigma,\Omega_k^h})$$
$$+ r^{-(\lambda-1)} M_\lambda (\|u - u_p^h\|_{1,\Omega_k^h} + \|u\|_{1,\Omega_k^h}) \} \|w_p^h\|_{1,\Omega_k^h}$$
$$\leq C \{ (r^{-(\lambda-1)} M_\lambda + M_{0,\infty}) (\|u - u_p^h\|_{1,\Omega} + q^{-(\sigma-1)} h^{(\sigma-1)} \|u\|_{\sigma,\Omega})$$
$$+ r^{-(\lambda-1)} M_\lambda \|u\|_{1,\Omega} \} \|w_p^h\|_{1,\Omega}.$$

The Lemma follows from dividing by $\|w_p^h\|_{1,\Omega}$. □

By a direct application of Lemma 3.3 and (1.1) to Lemma 3.1 we obtain the following main Theorem which gives an asymptotic, $H^1(\Omega)$-norm estimate for the rate of convergence under numerical quadrature rules

THEOREM 3.4. *Let $I_m \in G_p$ be a quadrature rule defined on $\widehat{\Omega} \subset R^2$, which satisfies $d(m)-p-1>0$. We assume that $u \in H^\sigma(\Omega), a \in H^\alpha(\Omega)$ and $\widehat{a}_{ij} \in H^\rho(\widehat{\Omega})$ for $i,j=1,2$ such that $\lambda = \min(\alpha,\rho) \geq 2$. Then, for any positive integer q such that $0<q\leq p$, we have*

(3.28) $$\|u-\widetilde{u}_p^h\|_{1,\Omega}$$
$$\leq C\{(r^{-(\lambda-1)}M_\lambda + M_{0,\infty})q^{-(\sigma-1)}h^\mu\|u\|_{\sigma,\Omega} + r^{-(\lambda-1)}M_\lambda\|u\|_{1,\Omega}\},$$
where $\mu=\min(p,\sigma-1)$ and $r=d(m)-p-q$.

proof. Taking $v_p^h \in S_{p,0}^h(\Omega)$ with an approximation u_p^h of u which satisfies (2.17), we obtain from Lemma 3.1 that

(3.29) $$\|u-\widetilde{u}_p^h\|_{1,\Omega}$$
$$\leq C\{\|u-u_p^h\|_{1,\Omega} + \sup_{w_p^h\in S_{p,0}^h(\Omega)} \frac{|B(u_p^h,w_p^h)-B_{m,\Omega}(u_p^h,w_p^h)|}{\|w_p^h\|_{1,\Omega}}\}.$$

Since $0<q\leq p$ it follows from (1.1) and Lemma 3.3 that the first term of the right side in (3.29) is dominated by its last term. Hence, the proof is completed by a direct application of Lemma 3.3 to (3.29).

□

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