Galois Groups of Subfactors

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Abstract

We review the subfactor theory in terms of bimodules and describe the Galois group of subfactors through 1 dimensional bimodules appearing in some level of derived tower. Some examples of the Galois groups are given in the case of fixed point algebras, crossed product algebras, and their compositions.

1 Introduction

After A. Ocneanu ([13]) has introduced a paragroup as an invariant for the classification of subfactors, the study of bimodules plays an important role in the theory of subfactors, initiated by V. Jones ([7]).

The paragroup for $\mathbf{N} \subset \mathbf{M}$ with finite Jones index is a grouplike object which provides information about the position of the subfactor $\mathbf{N}$ in a type II_1 factor $\mathbf{M}$. Indeed, a paragroup consists of a principal graph and a family of anti-automorphisms, which requires the study of bimodules and their intertwiners.

The first part of the paper is devoted to give an exposition on elementary theory of subfactors in terms of bimodules. Based on this notion, we show how the vertices of the principal graph is related to certain irreducible bimodules. Then we clarify the Galois groups for the inclusion by counting on the dimensions of intertwiners spaces in Section 3. In particular, we examine the result with various examples, like fixed point algebras, crossed product algebras, and their compositions.

It is worth to mention that there are several equivalent ways of describing paragroups, for example, Longo's sector theory ([11]) which is an analogous to bimodule theory. While properly infinite factor cases require sector theory naturally , we use a bimodule approach for the type II_1 factor cases. Among various materials about bimodules (sectors) associated to a subfactor, we refer [5, 8, 9, 10, 11, 15] for the references.

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2 Preliminaries

2.1 Bimodules

Let A and B be type II_1 factors. When B° denotes the opposite algebra of B, an A-B bimodule $X =_A X_B$ is defined as a Hilbert space X with normal *-representations of A and B° which commute each other, equipped with actions $a \cdot \xi \cdot b$, where $a \in A, b \in B$ and $\xi \in X$.

We say that two A-B bimodules X and Y are said to be unitary equivalent, denoted by $X \cong Y$, if there exists a surjective isometry $u : X \to Y$ commuting with the left and right actions, i.e. $u(a \cdot \xi \cdot b) = a \cdot u(\xi) \cdot b$. The categorical operations on the set of bimodules are given as follows.

1. The conjugate (contragredient) B-A bimodule ${}_BX^*_A$ of ${}_AX_B$ is the conjugate Hilbert space X^*, with the action $b \cdot \xi^* \cdot a = (a^* \cdot \xi \cdot b^*)^*$.
2. Let C be a type II_1 factor. Then the relative tensor product of A-B bimodule ${}_AX_B$ and B-C bimodule ${}_BY_C$ over B is given by the A-C bimodule ${}_AX \otimes_B Y_C$.

We have $(X^*)^* \cong X$ and $({}_AX \otimes_B Y_C)^* \cong_C Y^* \otimes_B X^*{}_A$. An A-A bimodule X is called self-conjugate if ${}_AX^*_A \cong_A X_A$.

The statistical dimension of the bimodule of X is defined as

$$d(X) = d({}_AX_B) = \sqrt{dim({}_AX) \cdot dim(X_B)}.$$

A bimodule X is said to have finite type if $d(X) < \infty$. It is clear that $d(X) \geq 0$ for nonzero bimodule X. Moreover, the dimension function d

on finite type bimodules satisfies algebraic properties; $d(X^*) = d(X)$, $d(X \oplus Y) = d(X)+d(Y)$ and $d(X \otimes Y) = d(X)d(Y)$. Also d is a conjugacy invariant function, i.e. when $X \cong Y$, we have $d(X) = d(Y)$.

Let $Hom(X, Y)$ be the set of intertwiners from A-B bimodules $X =_A X_B$ to $Y =_A Y_B$ which commute with right and left actions of A and B resp., that is, a set of bounded linear operator $T : X \to Y$ satisfying $T(a \cdot \xi \cdot b) = a \cdot T(\xi) \cdot b$. When $X = Y$, we denote by $End(X)$. Then $End(X) = End(_A X_B) \cong A' \cap End(X) \cap (B^\circ)'$, which is a von Neumann algebra on X. A bimodule X is called irreducible if $End(X) = CI_X$. When X is an A-B bimodule of finite type, X can be decomposed into finitely many direct sum of irreducible submodules as follows;

$$_A X_B \cong \sum \oplus m_i X_i \quad \text{with irreducible A-B bimodules } X_i,$$

where $m_i = dimHom(X, X_i) \neq 0$. Note that $X_i \cong pX$ for a minimal projection p on X.

We end this section with the following useful observation ([10, 15]). Let A be a type II_1 factor and $L^2(A)$ the Hilbert space by GNS completion of A with respect to the inner product defined by the trace. Then the normal representations of A and A° play the left and right actions which provide the bimodule structure to its Hilbert space $_A L^2(A)_A$ with actions

$$a \cdot \xi \cdot b = aJ_A b^* J_A \xi \ , \text{ for } a, b \in A, \xi \in L^2(A),$$

where J_A denotes the modular convolution of $L^2(A)$.

Lemma 2.1 *Let A and B be type II_1 factors and $_A X_B$ and $_A Y_B$ the A-B bimodules of finite type. Then the followings hold ;*

1. $Hom(X, Y) \cong Hom(Y^*, X^*)$
2. $Hom(X, Y) \cong Hom(_A L^2(A)_A, Y \otimes_B X^*)$

Theorem 2.2 (Frobenius reciprocity) *Let A, B and C be type II_1 factors and $_A X_B, _B Y_C, _A Z_C$ the bimodules of finite types. Then the following vector spaces are isomorphic ;*

$$Hom(_A X \otimes_B Y_C, _A Z_C) \cong Hom(_A Z \otimes_C Y^*_B, _A X_B) \cong Hom(_B X^* \otimes_A Z_C, _B Y_C).$$

Proof. It follows from Lemma 2.1, using the contragredient bimodules. □

We often use this fact in the sense that they have the same dimension over C.

2.2 Bimodules associated to $\mathbf{N} \subset \mathbf{M}$

Let $\mathbf{M}$ be a type II_1 factor. Let $L^2(\mathbf{M})$ be the Hilbert space by GNS completion of $\mathbf{M}$ with respect to the inner product defined by the trace. The normal representations of $\mathbf{M}$ and $\mathbf{M}^\circ$ play the left and right actions which provide the bimodule structure to its Hilbert space ${}_M L^2(\mathbf{M})_M$ with actions

$$n \cdot \xi \cdot m = nJ_M m^* J_M \xi \ , \ \text{for } n, m \in \mathbf{M}, \xi \in L^2(\mathbf{M}),$$

where J_M denotes the modular convolution of $L^2(\mathbf{M})$.

For a given inclusion of factors $\mathbf{N} \subset \mathbf{M}$, we now associate a $\mathbf{N}$-$\mathbf{M}$ bimodule. By restricting left action to $\mathbf{N}$, the associated bimodule is defined by $H \stackrel{\text{let}}{=} {}_N H_M = {}_N L^2(\mathbf{M})_M$, which becomes a $\mathbf{N}$-$\mathbf{M}$ bimodule with the actions $n \cdot \xi \cdot m$ ($n \in \mathbf{N}, m \in \mathbf{M}$). Then conjugate $\mathbf{M}$-$\mathbf{N}$ bimodule of H is given by $H^* = {}_M L^2(\mathbf{M})^*_N \cong {}_M L^2(\mathbf{M})_N = {}_M H_N$, i.e. the same Hilbert space $L^2(\mathbf{M})$ with restricted right action to $\mathbf{N}$.

The Jones's index of $\mathbf{N} \subset \mathbf{M}$ is defined by $[\mathbf{M} : \mathbf{N}] = dim_N H$, the coupling constant of $\mathbf{N}$ on the Hilbert space $L^2(\mathbf{M})$. When $[\mathbf{M} : \mathbf{N}] < \infty$, the statistical dimension of the $\mathbf{N}$-$\mathbf{M}$ bimodule H is

$$d(H) = d({}_N H_M) = \sqrt{dim_N H \, dim H_M} = [\mathbf{M} : \mathbf{N}]^{\frac{1}{2}}.$$

Hence $\mathbf{N}$-$\mathbf{M}$ bimodule H is of finite type when $[\mathbf{M} : \mathbf{N}]$ is finite. Moreover the fact of $End(H) \cong \mathcal{B}(L^2(H))$ gives that

$$End({}_N H_M) = \mathbf{N}' \bigcap (\mathbf{M}^\circ)' \bigcap End(H) = \mathbf{N}' \bigcap \mathbf{M},$$

and so H is an irreducible $\mathbf{M}$-$\mathbf{N}$ bimodule if and only if $\mathbf{N}$ is an irreducible subfactor of $\mathbf{M}$.

2.3 The graph invariants

For a pair of factors $\mathbf{N} \subset \mathbf{M}$, the basic construction $\mathbf{M}_1$ for $\mathbf{N} \subset \mathbf{M}$ is defined as $\mathbf{M}_1 = End(H_N)$, the algebra of endomorphisms of $\mathbf{M}$ viewed as a right $\mathbf{N}$-module. Indeed,

$$\mathbf{M}_1 = (\mathbf{N}^\circ)' \cap End(H) = J_M \mathbf{N}' J_M \supset J_M \mathbf{M}' J_M = \mathbf{M}.$$

When $[\mathbf{M}:\mathbf{N}]<\infty$, $\mathbf{M}_1$ is also type II_1 factor with $[\mathbf{M}_1:\mathbf{M}]=[\mathbf{M}:\mathbf{N}]$. This fact enables us to iterate the basic constructions to obtain Jones tower of II_1 factors;

$$\mathbf{N}=\mathbf{M}_{-1}\subset\mathbf{M}=\mathbf{M}_0\subset\mathbf{M}_1\subset\mathbf{M}_2\subset\cdots.$$

The derived tower is an increasing sequence of relative commutants algebras

$$\mathcal{C}I=\mathbf{N}'\cap\mathbf{N}\subset\mathbf{N}'\cap\mathbf{M}\subset\mathbf{N}'\cap\mathbf{M}_1\subset\mathbf{N}'\cap\mathbf{M}_2\subset\cdots.$$

The following observation provides the bimodule description of the derived tower. Note that $H\otimes H^*=L^2(\mathbf{M})\otimes_M L^2(\mathbf{M})={}_N L^2(\mathbf{M})_N$, $\mathbf{N}$-$\mathbf{N}$ bimodule with left and right actions restricted to $\mathbf{N}$. Due to the fact $\mathbf{M}_1=L^2(\mathbf{M})\otimes_N L^2(\mathbf{M})$, we also see that

$$H\otimes H^*\otimes H={}_N\, L^2(\mathbf{M})\otimes_N L^2(\mathbf{M})_M={}_N\, L^2(\mathbf{M}_1)_M.$$

Inductively, we have

$${}_N L^2(\mathbf{M}_k)_N=(H\otimes_N H^*)^k,\quad \text{as } \mathbf{N}\text{-}\mathbf{N} \text{ bimodues},$$

$${}_N L^2(\mathbf{M}_k)_M=(H\otimes_N H^*)^k\otimes_N H,\quad \text{as } \mathbf{N}\text{-}\mathbf{M} \text{ bimodules}.$$

Proposition 2.3 *Let* $\mathbf{N}=\mathbf{M}_{-1}\subset\mathbf{M}=\mathbf{M}_0\subset\mathbf{M}_1\subset\mathbf{M}_2\subset\cdots$ *be the Jones tower. Then the derived tower is given by the spaces of* $\mathbf{N}$-$\mathbf{N}$ *and* $\mathbf{N}$-$\mathbf{M}$ *bimodule intertwiners. More precisely, for* $k\geq 0$,

$$\mathbf{N}'\bigcap\mathbf{M}_{2k+1}=End({}_N L^2(\mathbf{M}_k)_N)=End((H\otimes H^*)^k),$$

$$\mathbf{N}'\bigcap\mathbf{M}_{2k}=End({}_N L^2(\mathbf{M}_k)_M)=End((H\otimes H^*)^k\otimes H).$$

Proof. Since $\mathbf{N}\subset\mathbf{M}_k\subset\mathbf{M}_{2k+1}$ and $\mathbf{M}\subset\mathbf{M}_k\subset\mathbf{M}_{2k}$ are basic constructions, we have $\mathbf{M}_{2k+1}=End(L^2(\mathbf{M}_k)_N)$ and $\mathbf{M}_{2k}=End(L^2(\mathbf{M}_k))_M)$. With help of above observation, we have the results. □

Notice that $\mathbf{N}'\cap\mathbf{M}_k$ are finite dimensional C^*-algebras when $d(H)<\infty$. So the space of intertwiners is isomorphic to a multimatrix algebra as long as Jones index is finite. In this case, each direct summand matrix algebra appearing in $\mathbf{N}'\cap\mathbf{M}_k$ is isomorphic to an irreducible $\mathbf{N}$-$\mathbf{N}$ or $\mathbf{N}$-$\mathbf{M}$ bimodule.

The principal graph of $\mathbf{N} \subset \mathbf{M}$ is defined by the Bratteli diagram of the collection of equivalent classes of irreducible $\mathbf{N}$-$\mathbf{N}$ bimodules $\{X\}$, $\mathbf{N}$-$\mathbf{M}$ bimodules $\{Y\}$ appearing as components in $End((H \otimes H^*)^k)$ and $End((H \otimes H^*)^k \otimes H)$, respectively. More explicitly, the principal graph of $\mathbf{N} \subset \mathbf{M}$ is the bipartite graph with even vertices indexed by $\mathbf{N}$-$\mathbf{N}$ irreducible bimodules $\{X\}$ and odd vertices indexed by $\mathbf{N}$-$\mathbf{M}$ irreducible bimodules $\{Y\}$, and $dimHom(X \otimes H, Y)$ edges between a vertex X and a vertex Y.

When H is irreducible, the principal graph is connected. If the graph is finite, then $\mathbf{N} \subset \mathbf{M}$ is said to have finite depth. Otherwise, it is said to have infinite depth. In other words, $\mathbf{N} \subset \mathbf{M}$ has finite depth when there are only finitely many equivalent classes of irreducible $\mathbf{N}$-$\mathbf{N}$, $\mathbf{N}$-$\mathbf{M}$ bimodules appearing in the decompositions of the derived tower.

The depth of $\mathbf{N} \subset \mathbf{M}$ is defined by the longest distance from the vertex indexed by $\mathbf{N}' \cap \mathbf{N} = End({}_N L^2(\mathbf{N})_N)$ to every vertices of the principal graph.

Remark 2.4 *We can also proceed the same methods to obtain dual principal graph, starting with the associated* $\mathbf{M} - \mathbf{N}$ *bimodule* $H^* = {}_M L^2(\mathbf{M})_N$. *Then*

$$\mathbf{M}' \cap \mathbf{M}_1 = End(H^*) \ \ as\ \mathbf{M}\text{-}\mathbf{N}\ bimodules,$$

$$\mathbf{M}' \cap \mathbf{M}_2 = End(H^* \otimes H) \ \ as\ \mathbf{M}\text{-}\mathbf{M}\ bimodules,$$

$$\mathbf{M}' \cap \mathbf{M}_3 = End(H^* \otimes H \otimes H^*) \ \ as\ \mathbf{M}\text{-}\mathbf{N}\ bimodules,$$

and so on.

3 Applications

3.1 Depth two subfactors

In this section, we describe the well-known structure of depth two subfactors $\mathbf{N} \subset \mathbf{M}$ in bimodule interpretation.

Definition 3.1 *An irreducible subfactor* $\mathbf{N} \subset \mathbf{M}$ *has depth two if* $\mathbf{N}' \cap \mathbf{M}_2 = End(H \otimes H^* \otimes H)$ *is a factor.*

It is equivalent to $dim(\mathbf{N}' \cap \mathbf{M}_1) = dimEnd(H \otimes H^*) = [\mathbf{M} : \mathbf{N}]$ ([14]).

Lemma 3.2 ([5, 12]) *Let* $[\mathbf{M}:\mathbf{N}]<\infty$ *and* $\mathbf{N}'\cap\mathbf{M}=CI$. *If* $H\otimes H^*\cong\sum\oplus m_i\cdot X_i$ *where* $X_i's$ *are irreducible* **N-N** *submodules with* $m_i = dimHom(X_i, H\otimes H^*)$, *then the followings are equivalent;*

1. $\mathbf{N}\subset\mathbf{M}$ *have depth 2,*
2. $d(X_i)=m_i$.

Proof. Notice that H is an irreducible **N-M** binmodule due to $\mathbf{N}'\cap\mathbf{M}=CI$.
($1\Rightarrow 2$) Let $\mathbf{N}\subset\mathbf{M}$ have depth two. Then $End(H\otimes H^*\otimes H)$ is a factor. Since $H\otimes H^*\otimes H$ contains H as an irreducible component, $H\otimes H^*\otimes H = [\mathbf{M}:\mathbf{N}]H$ with help of the multiplicative property of dimension function d.

On the other hand, $[\mathbf{M}:\mathbf{N}]H=(H\otimes H^*)\otimes H=\sum\oplus m_i(X_i\otimes H)$, and so we must have $X_i\otimes H=kH$ for some k. But Frobenius reciprocity implies that

$$m_i = dimHom(X_i, H\otimes H^*) = dimHom(X_i\otimes H, H) = dimHom(kH, H) = k.$$

Hence we have $X_i\otimes H=m_iH$. With this fact, the multiplicative property of the function d again gives $d(X_i)=m_i$.
($2\Rightarrow 1$) Let $d(X_i)=m_i$. Since $[\mathbf{M}:\mathbf{N}]=d(H\otimes H^*)=\sum m_i d(X_i)=\sum m_i^2$,

$$\begin{aligned}
dimEnd(H\otimes H^*) &= dimHom(H\otimes H^*\otimes H, H)\\
&= \sum m_i\, dimHom(X_i\otimes H, H)\\
&= \sum m_i\, dimHom(X_i, H\otimes H^*)\\
&= \sum m_i^2 = [\mathbf{M}:\mathbf{N}].
\end{aligned}$$

This completes the proof. □

Theorem 3.3 *Let* $\mathbf{N}\subset\mathbf{M}$ *have depth two. Then* $A=End(H\otimes H^*)$ *is a* $[\mathbf{M}:\mathbf{N}]$ *dimensional Hopf algebra acting outerly on* $\mathbf{M}$ *such that*

$$\mathbf{M}_1=\mathbf{M}\times A \quad or \quad \mathbf{M}=\mathbf{N}\times A^\circ,$$

where $A^\circ = End(H^*\otimes H)$ *denotes the dual Hopf algebra of* A.

Proof. See [12, 14]. □

Remark 3.4 *There is always a canonical pairing between* $End(H\otimes H^*)$ *and* $End(H^*\otimes H)$, *thanks to Frobenius reciprocity. In the case of depth two, this pairing provides Hopf algebra structures to* $End(H\otimes H^*)$, *and its dual* $End(H^*\otimes H)$ *as well.*

4 Galois groups of subfactors

Let $\mathbf{N} \subset \mathbf{M}$ be factors of finite index with $\mathbf{N}' \cap \mathbf{M} = \mathcal{C}I$. The Galois group for the pair is defined by

$$Gal(\mathbf{M}, \mathbf{N}) = \{\alpha \in Aut(\mathbf{M}) \mid \alpha \mid_N = id \mid_N\}.$$

When $\mathbf{M}_1$ denotes the basic construction for $\mathbf{N} \subset \mathbf{M}$, the set of unitaries in $\mathbf{M}_1$ normalizing $\mathbf{M}$ is denoted by $\mathcal{N}_M(\mathbf{M}_1) = \{u \in \mathcal{U}(\mathbf{M}_1) \mid u\mathbf{M}u^* = \mathbf{M}\}$. Recall (for example, see [2]) that

1. $Gal(\mathbf{M}, \mathbf{N})$ is a group of order less than or equal to $[\mathbf{M} : \mathbf{N}]$,

2. $Gal(\mathbf{M}, \mathbf{N})$ is isomorphic to $\mathcal{N}_M(\mathbf{M}_1)/\mathcal{U}(\mathbf{M})$.

In this section we focus on computation of Galois groups based on the results in Chapter 3. For thsi we first discuss one dimensional **M**-**M** bimodules, and then relate the Galois group to one dimensional **M**-**M**bimodules appearing at some level of derived tower.

4.1 Automorphisms in bimodules

Recall that the standard Hilbert bimodule $H = {}_ML^2(\mathbf{M})_M$ has dimemsion 1, equipped with the action $m_1 \cdot \xi \cdot m_2 = m_1 J_M m_2^* J_M \xi$ for $m_1, m_2 \in \mathbf{M}$ and $\xi \in L^2(\mathbf{M})$. This provides a multiplicative unit for the relative tensor product.

Let α be an automorphism of **M**. Then α gives rise to another **M**-**M** bimodule ${}_M({}^\alpha H)_M$ (resp. ${}_M(H^\alpha)_M$) of dimension 1, which is the same Hilbert space $L^2(\mathbf{M})$ equipped with the action

$$m_1 \cdot \xi \cdot m_2 = \alpha(m_1) J_M m_2^* J_M \xi,$$

$$(\text{resp.,} \quad m_1 \cdot \xi \cdot m_2 = m_1 J_M \alpha(m_2)^* J_M \xi),$$

where $m_1, m_2 \in \mathbf{M}$.

It is clear that ${}^\alpha H \cong {}^\beta H$ if and only if there is a unitary $u \in \mathbf{M}$ such that $\alpha = (Adu)\beta$ on $\mathbf{M}$, for $\alpha, \beta \in Aut(\mathbf{M})$.

Lemma 4.1 ([5, 8]) *Let* $\mathbf{M}$ *be a type* II_1 *factor and* $\alpha, \beta \in Aut(\mathbf{M})$. *Then we have the following ;*

1. ${}^{\alpha}H \cong H^{\alpha^{-1}}$,
2. $({}^{\alpha}H)^* \cong H^{\alpha}$,
3. ${}^{\alpha}H \otimes_M {}^{\beta}H \cong {}^{\beta\alpha}H,\ H^{\alpha} \otimes_M H^{\beta} \cong H^{\alpha\beta}$,
4. ${}^{\alpha}H \cong H$ *if and only if* $\alpha \in Inn(\mathbf{M})$.

Note that Lemma 4.1 implies that ${}^{\alpha}H$ is self-conjugate if and only if $\alpha^2 \in Inn(\mathbf{M})$. Finally we have the following ([10, 9]).

Proposition 4.2 *A bimodule* ${}_MX_M$ *has* $d(X) = 1$ *if and only if there exists an outer automorphism* $\alpha \in Aut(\mathbf{M})$ *so that* $X \cong {}^{\alpha}H$.

Proof. For $x \in X$, let $m \cdot x = \pi_l^X(m)x$, $x \cdot m = \pi_r^X(m)x$ be the left and right representations of $\mathbf{M}$ on X, while $\mathbf{M}$ has a standard form on $L^2(\mathbf{M})$, with left and right multiplications $m \cdot \xi = \pi_l(m)\xi$, $\xi \cdot m = \pi_r(m)\xi$ respectively, for $\xi \in L^2(\mathbf{M})$.

It is enough to show the necessity. Assume that ${}_MX_M$ has $d({}_MX_M) = 1$. Then the coupling constant $dim_M X = dim X_M = 1$, and hence $\mathbf{M}$ has a standard form on X as well. Therefore there is a unitary $u : X \to L^2(\mathbf{M})$ such that $\pi_r(m) = u\pi_r^X(m)u^*$, for $m \in \mathbf{M}$.

Taking $\alpha = \pi_l^{-1} \circ Adu \circ \pi_l^X$, we see that $\alpha \in Aut(\mathbf{M})$ and $\pi_l(\alpha(m)) = u\pi_l^X(m)u^*$. Moreover, if $x \in X$,

$$u(m \cdot x \cdot n) = u\pi_l^X(m)\pi_r^X(n)x = \pi_l(\alpha(m))uu^*\pi_r(n)ux = \alpha(m)J_M n^* J_M \cdot ux.$$

Therefore, via u, $X \cong {}^{\alpha}H$ as $\mathbf{M}$-$\mathbf{M}$ bimodules. It now follows from Lemma 4.1 that α is not inner. □

4.2 Galois groups and grouplike elements

Let $\mathbf{N} \subset \mathbf{M}$ have depth two. Then $End(H^* \otimes H)$ is a $[\mathbf{M} : \mathbf{N}]$ dimensional Hopf algebra. In this section, we study irreducible $\mathbf{M}$-$\mathbf{M}$ bimodules of dimension 1 appearing in the Hopf algebra $End(H^* \otimes H)$. We call them the grouplike elements.

Notice that if two irreducible A-B bimodules X and Y satisfy $Hom(X, Y) \neq 0$, then $X \cong Y$ as A-B bimodules.

Proposition 4.3 ([10]) *Let* $\mathbf{N}' \cap \mathbf{M} = \mathcal{C}I$. *Then the set of the grouplike elements in* $End(H^* \otimes H)$ *forms a group under relative tensor product, with identity* ${}_M H_M$. *Moreover this group is isomorphic to the Galois group* $\mathcal{G}(\mathbf{M}, \mathbf{N})$.

Proof. If follows from Proposition 4.2 that there exists outer automorphism $\alpha \in Aut(\mathbf{M})$ corresponding to a 1 dimensional **M**-**M** submodule appearing in $End(H^* \otimes H)$. Then the group structure on the set of grouplike elements comes from Lemma 4.1 and Proposition 4.2.

Note that Frobenius reciprocity implies

$$\begin{aligned} 0 &\neq Hom({}_M{}^{\alpha}H_M\,,\, {}_M H^* \otimes_N H_M) \cong Hom({}_N H \otimes_M {}^{\alpha}H_M\,,\, {}_N H_M) \\ &\cong Hom({}_N H \otimes_M H^{\alpha^{-1}}{}_M\,,\, {}_N H_M) \cong Hom({}_N{}^{\alpha}H_M\,,\, {}_N H_M), \end{aligned}$$

when ${}_N{}^{\alpha}H_M$ with restricted left action of α to $\mathbf{N}$. But $End({}_N{}^{\alpha}H_M) = \alpha(\mathbf{N})' \cap \mathbf{M} \cong \mathcal{C}$, we see that ${}_N{}^{\alpha}H_M$ is an irreducible **N**-**M** bimodule. Thus, as **N**-**M** bimodules, ${}_N{}^{\alpha}H_M \cong_N H_M$, via a unitary v. Since v commute with the left action, $v\alpha(n) = nv$ for $n \in \mathbf{N}$, and hence $(Adv)\alpha|_N = id_N$. This completes the proof. □

Consequently, the Galois group $\mathcal{G}(\mathbf{M}, \mathbf{N})$ is determined by 1 dimensional **M**-**M** submodules appearing in $End(H^* \otimes H)$. Note that the condition $\mathbf{N}' \cap \mathbf{M} = \mathcal{C}I$ in Proposition 4.3 is essential (for the counter example, see [10]).

We are now going to calculate the Galois groups in various cases ([3, 4, 6, 10]).

Example 4.4 Let a finite group G act outerly on a type II_1 factor $\mathbf{N}$ via α. Let $\mathbf{M}$ be the crossed product $\mathbf{M} = \mathbf{N} \times_\alpha G$. Then the associated **N**-**M** bimodule $H =_N L^2(\mathbf{M})_M$ is irreducible, due to the outerness of the action. Moreover, the relative commutants are easily expressed as

$${}_N H \otimes_M H^*_N =_N L^2(\mathbf{M})_N = \sum \oplus_{g \in G} {}^{\alpha_g} L^2(\mathbf{N}),$$

with mutually non-equivalent irreducible **N**-**N** bimodules ${}^{\alpha_g}L^2(\mathbf{N})$ of dimension 1.

On the other hand,

$${}_M H^* \otimes_N H_M =_M L^2(\mathbf{M}_1)_M = \sum_{\sigma \in G^*, j} \oplus X_j^{\sigma},$$

where G^* denotes the set of representations of G. Here **M**-**M** bimodule $X^\sigma_j = \sum_i \oplus L^2(\mathbf{M})\xi^\sigma_{i,j}$ is an irreducible **M**-**M** bimodule with linear basis defined by $\xi^\sigma_{i,j} = \sum_{g\in G}\sigma_{i,j}(g^{-1})u_{g^{-1}} \otimes u_g \in H^* \otimes H$. Therefore X^σ_j is one dimensional bimodule if and only if σ is one dimesional representation of G, so the Galois group $\mathcal{G}(\mathbf{M},\mathbf{N})$ is group-isomorphic to the character group $\hat{G} \cong G/[G,G]$.

Example 4.5 Let $\mathbf{N} = \mathbf{M}^G \subset \mathbf{M}$. Then $H =_N L^2(\mathbf{M})_M$ is also irreducible bimodule and

$$_N H \otimes H^*_N =_{M^G} L^2(\mathbf{M})_{M^G} = \sum \oplus_{\sigma\in G^*} X^\sigma,$$

where G^* denotes the set of representations of G. Also

$$_M H^* \otimes H_M =_M L^2(\mathbf{M}_1)_M =_M L^2(\mathbf{M} \times_\alpha G)_M = \sum_{g\in G} \oplus\, {}^{\alpha_g}L^2(\mathbf{M}).$$

We see that each element $g \in G$ is associated with a 1 dimensional bimodule ${}^{\alpha_g}L^2(\mathbf{M})$, therefore Galois group $\mathcal{G}(\mathbf{M},\mathbf{N})$ is group-isomorphic to the group G.

Example 4.6 Let finite groups $K \subset G$ act outerly on a type II_1 factor **P**. We consider $\mathbf{N} = \mathbf{P} \times K \subset \mathbf{P} \times G = \mathbf{M}$. Let $s = KsK$ be the K-K double coset representatives in $K\backslash G/K$. If we let $K_s = K \cap sKs^{-1}$, then

$$_N H \otimes H^*_N =_{P\times K} L^2(\mathbf{P} \times \mathbf{G})_{P\times K} = \sum_s \sum_{i\in \hat{K}_s} \oplus Y^i_s,$$

where i denotes the identity representation of K_s. On the other hand, if we denote $\overline{G} = \{\sigma \in G^* \,|\, \sigma|_K = i_K\}$,

$$_M H^* \otimes H_M =_{P\times G} L^2(\mathbf{M}_1)_{P\times G} = \sum_{\sigma\in\overline{G}} \oplus X_\sigma.$$

Since X_σ is one dimensional if and only if $dim(\sigma) = 1$, the Galois group is isomorphic to the characters of G vanishing on K.

Example 4.7 Under the same condition as Example 4.6, we now consider $\mathbf{N} = \mathbf{P}^G \subset \mathbf{P}^K = \mathbf{M}$. Then the role of **N**-**N**, **M**-**M** bimodules are switched in this case. i.e.,

$$_N H \otimes H^*_N =_{P^G} L^2(\mathbf{P}^K)_{P^G} = \sum_{\sigma\in\overline{G}} \oplus X_\sigma,$$

and

$$_M H^* \otimes H_M =_{PK} L^2(\mathbf{M}_1)_{PK} = \sum_s \sum_{i \in \hat{K}_s} \oplus Y_s^i,$$

where i denotes the identity representation of K_s. Then irreducible bimodule Y_s^i has dimension one if and only if $K = K_s$, or $s \in \mathcal{N}_G(K)$; in other words, those K_s satisfying $K = K_s$ are paramatized by $\mathcal{N}_G(K)/K$. Thus the Galois group is isomorphic to $\mathcal{N}_G(K)/K$.

Example 4.8 Let H and K be subgroups in a finite group G such that $H \cap K = \{e\}$ and HK be a group. Let α and β be the outer actions of H and K respectively on a type II_1 factor $\mathbf{P}$. We consider the subfactors $\mathbf{N} = \mathbf{P}^H \subset \mathbf{P} \times_\alpha K = \mathbf{M}$. We know that $\mathbf{N} \subset \mathbf{M}$ is irreducible and have depth two in this case ([1, 3, 6]).

First, let $A =_N L^2(\mathbf{P})_P$, $B =_P L^2(\mathbf{M})_M$, and $X = A \otimes_P B =_N L^2(\mathbf{M})_M$ be the associated irreducible **N-M** bimodule. Note that

$$A^* \otimes A =_P L^2(\mathbf{P}) \otimes_N L^2(\mathbf{P})_P =_P L^2(\mathbf{P} \times_\beta H)_P = \sum \oplus_{h \in H} [\beta_h],$$

and

$$B \otimes B^* =_P L^2(\mathbf{M}) \otimes_M L^2(\mathbf{M})_P =_P L^2(\mathbf{P} \times_\alpha K)_P = \sum \oplus_{s \in K} [\alpha_s],$$

where $[\theta] =_P {}^\theta L^2(\mathbf{P})_P$ with the action $m \cdot \xi \cdot n = \theta(m) J_P n^* J_P \xi$ for $n, m \in \mathbf{P}$, and an outer automorphism θ of $\mathbf{P}$.

For the Galois group, we need to compute the submodules of dimension one appearing in $X^* \otimes X$. Since

$$\begin{aligned} X^* \otimes X &= {}_M(B^* \otimes_P A^*) \otimes_N (A \otimes B)_M \\ &= {}_M B^* \otimes_P (\sum \oplus_{h \in H} [\beta_h]) \otimes_P B_M \\ &= \sum \oplus_{h \in H} \; {}_M(B^* \otimes_P [\beta_h] \otimes_P B)_M, \end{aligned}$$

we have

$$\begin{aligned} End(X^* \otimes X) &= Hom(X^* \otimes X, X^* \otimes X) \\ &= \sum \oplus_{g,h \in H} Hom(B^* \otimes_P [\beta_g] \otimes_P B, B^* \otimes_P [\beta_h] \otimes_P B). \end{aligned}$$

On the other hand, for $g, h \in H$,

$$Hom(B^*\otimes_P[\beta_g]\otimes_P B, B^*\otimes_P[\beta_h]\otimes_P B) = Hom([\beta_g], B\otimes_P B^*\otimes_P[\beta_h]\otimes_P B\otimes_P B^*)$$
$$= \sum \oplus_{s,t\in K} Hom([\beta_g], [\alpha_s]\otimes[\beta_h]\otimes[\alpha_t]).$$

But $Hom([\beta_g], [\alpha_s]\otimes[\beta_h]\otimes[\alpha_t]) = 0$ for some $s, t \in K$ if and only if $g \neq sht$ in $Out(\mathbf{P})$. i.e, $B^*\otimes_P[\beta_g]\otimes_P B$ and $B^*\otimes_P[\beta_h]\otimes_P B$ are disjoint if and only if $KgK \neq KhK$. In other words, the double cosets KhK of $K\backslash G/K$ parameterize nonequivalent bimodules $B^*\otimes_P[\beta_h]\otimes_P B$. Therefore we have

$$End(X^*\otimes X) = \sum_{h=KhK} \oplus End(B^*\otimes_P[\beta_h]\otimes_P B)$$

$$= \sum_{h=KhK}\sum_{s,t\in K} \oplus Hom([\beta_h], [\alpha_s]\otimes[\beta_h]\otimes[\alpha_t]),$$

where $h = KhK \in K\backslash G/K$ is the representative of K-K double cosets in G. But $Hom([\beta_h], [\alpha_s]\otimes[\beta_h]\otimes[\alpha_t]) \neq 0$ if and only if $ths = h$, or $t = hs^{-1}h^{-1} \in hKh^{-1}\cap K$. Therefore

$$End(X^*\otimes X) = \sum_{h=KhK}\sum_{s\in K\cap hKh^{-1}} \oplus Hom([\beta_h], [\alpha_s]\otimes[\beta_h]\otimes[\alpha_{hs^{-1}h^{-1}}])$$

$$\cong \sum_{h=KhK} \oplus \mathcal{C}_\omega[K\cap hKh^{-1}],$$

the direct sum of twisted group algebras. Therefore the Galois group is determined by one dimensional irreducible representations of $K_h = K\cap hKh^{-1}$, for all $h \in K\backslash G/K$.

Lemma 4.9 *$\chi \in \hat{K_h}$ has dimension 1 if and only if $h \in \mathcal{N}_H(K)$, where $\mathcal{N}_H(K) = \{h \in H \mid hKh^{-1} = K\}$ denotes the normalizer subgroup of K in H.*

Proof. For $\chi \in \hat{K_h}$, note that

$$\frac{dim\chi\cdot|KhK|}{|K|^2} = (dim\chi)^2\cdot\frac{|K|}{|K_h|}.$$

Since $|KhK| = |K|^2/|K_h|$, $dim\chi = \frac{|K_h|}{|K|}$. Thus $dim\chi = 1$ if and only if $|K| = |K_h|$, if and only if $h \in H$ satisfies $hKh^{-1} = K$, or $h \in \mathcal{N}_H(K)$. □

With Lemma 3.9, we finally have

$$\mathcal{G}(\mathbf{M},\mathbf{N}) = \sum_{h \in \mathcal{N}_H(K)} \oplus \mathcal{C}_\omega(K).$$

Now the mapping of $h \in \mathcal{N}_H(K)$ into a map $(\chi \longrightarrow \chi^h)$ determines an action of $\mathcal{N}_H(K)$ on $\hat{K}$, where $\chi^h(k) = \chi(h^{-1}kh)$ for $\chi \in \hat{K}$. Hence the Galois group $\mathcal{G}(\mathbf{M},\mathbf{N})$ is isomorphic to the semidirect product group $\hat{K} \times \mathcal{N}_H(K)$.

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