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Fuzzy Pseudo-Topological Structure

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I. Introduction and preliminaries

In 1964, D. C. Kent [5] introduced the convergence function and investigated some properties. Using this notion, we defined in [6] a fuzzy convergence structure which was a correspondence between the prefilters on a given set X and the fuzzy subsets of X and showed that, with each fuzzy convergence structure, there was an associated fuzzy pretopological structure, that is the finest fuzzy pretopological structure which is coarser than the fuzzy convergence structure. In [7], we introduced the notion of fuzzy initial convergence structure and investigated some properties.

In this paper, we introduce concepts of fuzzy pseudo-topological structure and fuzzy pseudo-topologically coherent and investigate some properties with respect to the initial convergence structure.

We recall some definitions and known results to be used in sequel (See [2, 3, 6, 7])

Let X be a nonempty set and I be the unit interval. A *fuzzy set* in X is an element of the set of all functions from X into I. For each fuzzy set A and B in X, $A \subseteq B$ if $A(x) \leq B(x)$ for all $x \in X$. A *fuzzy point* p in X is fuzzy set in X defined by $p(x) = \lambda$ for $x = x_p$ $(0 < \lambda \leq 1)$ and $p(x) = 0$ for $x \neq x_p$. We call x_p

the *support* of p and λ the *value* of p. A fuzzy point $p \in A$, where A is a fuzzy set in X, if $p(x_p) \leq A(x_p)$. A fuzzy point p is said to be *quasi-coincident* with A, denoted by pqA, if $p(x_p)+A(x_p)>1$ or $p(x_p)>A'(x_p)$, where A' is the complement of A, denoted by $A'=1-A$.

For a nonempty set X, $F(X)$ denotes the set of all prefilters on X and $P(X)$ the set of all fuzzy sets on X. For each fuzzy point p in X, $\dot{p}$ is denoted by $\{A \in P(X) \mid p \, q A\}$.

A *fuzzy convergence structure* on X is a function c from $F(X)$ into $P(X)$ satisfying the following conditions :

(1) For each fuzzy point p in X, $p \in c(\dot{p})$.

(2) For Φ, $\Psi \in F(X)$, if $\Phi \subseteq \Psi$, then $c(\Phi) \subseteq c(\Psi)$.

(3) If $p \in c(\Phi)$, then $p \in c(\Phi \cap \dot{p})$.

The pair (X, c) is said to be *fuzzy convergence space*. If $p \in c(\Phi)$, we say that Φ *c-converges* to p. The filter $V_c(p)$ obtained by intersecting all prefilters which c-converges to p is said to be the *c-neighborhood prefilter* at p. If $V_c(p)$ c-converges to p for each fuzzy point p in X, then c is said to be a *fuzzy pretopological structure*, and (X, c) a *fuzzy pretopological space*. The fuzzy pretopological structure c is said to be *fuzzy topological structure* and (X, c) is said to be *fuzzy topological space*, if for each fuzzy point p in X, the prefilter $V_c(p)$ has a prefilterbase $B_c(p) \subseteq V_c(p)$ with the following property :

$$r\, q\, U \in \boldsymbol{B}_c(p) \text{ implies } U \in \boldsymbol{B}_c(r).$$

A fuzzy convergence space (X, c) is said to be *fuzzy compact* if every ultraprefilter c-converges. And $\alpha_c(\Phi)$ is defined as follows:

$$\alpha_c(\Phi) = \{p \mid p \in c(\Psi), \text{ for any prefilter } \Psi \text{ finer than } \Phi\}.$$

Theorem 1.1([6]). Let $C(X)$ be the set of all fuzzy convergence structures on X and $\leq$ a relation on $C(X)$ defined as follows:

$$c_1 \leq c_2 \text{ if and only if } c_2(\Phi) \subseteq c_1(\Phi)$$

for each $c_1, c_2 \in C(X)$, $\Phi \in F(X)$. Then $\leq$ is a partially order on $C(X)$.

Let $c_1, c_2 \in C(X)$. If $c_1 \leq c_2$, we say that c_2 is *finer* than c_1, also that c_1 is *coarser* than c_2.

Theorem 1.2([6]). Let c be a fuzzy convergence structure on X, and $\hat{c}$ a function from $F(X)$ to $P(X)$ defined by

$$p \in \hat{c}(\Phi) \text{ if and only if } \boldsymbol{V}_c(p) \subseteq \Phi$$

for each $\Phi \in F(X)$. Then $\hat{c}$ is the finest fuzzy pretopological structure on X coarser than c.

Let f be a function from a fuzzy convergence space (X, c_X) onto a fuzzy convergence space (Y, c_Y). Then f is said to be *fuzzy continuous* at a fuzzy point

p if the prefilter $f(\Phi)$ c_Y-converges to $f(p)$ for every prefilter Φ c_X-converging to p. If f is fuzzy continuous at every fuzzy point p in X, then f is said to be *fuzzy continuous*.

Theorem 1.3([6]). Let $f: (X, c_X) \to (Y, c_Y)$ be a function.

1) If f is fuzzy continuous at fuzzy point p in X, then

$$\boldsymbol{V}_{c_Y}(f(p)) \subset f(\boldsymbol{V}_{c_X}(p)).$$

2) If c_Y\$c_Y\$ is a fuzzy pretopological structure and

$$\boldsymbol{V}_{c_Y}(f(p)) \subset f(\boldsymbol{V}_{c_X}(p)),$$

then f is fuzzy continuous at fuzzy point p.

Let $\{X_i \mid i \in I\}$ be a family of sets, $\boldsymbol{B}_i$ a prefilterbase on X_i for each $i \in I$ and $\boldsymbol{B}$ the family of fuzzy sets in the product set $\prod_{i \in I} X_i$ which are of the form $\prod_{i \in I} B_i$, where $B_i = X_i$ except for a finite number of induces and $B_i \in \boldsymbol{B}_i$ for each $i \in I$ such that $B_i \neq X_i$. The formula

$$(\prod_{i \in I} B_i) \cap (\prod_{i \in I} C_i) = \prod_{i \in I} (B_i \cap C_i)$$

shows that $\boldsymbol{B}$ is a fuzzy prefilterbase on $\prod_{i \in I} X_i$.

Let Φ_i be a prefilter on X_i for each $i \in I$. The product of $\{\Phi_i \mid i \in I\}$ is the prefilter on $\prod_{i \in I} X_i$ generated by the family of the form $\prod_{i \in I} B_i$, where $B_i \in \Phi_i$ for each $i \in I$ and $B_i = X_i$ for all but a finite number of induces. The product prefilter of $\{\Phi_i \mid i \in I\}$ is denoted by $\prod_{i \in I} \Phi_i$.

Let (X_i, c_i) be a fuzzy convergence space for each $i \in I$, $f_i : X \to (X_i, c_i)$ be a surjection and c be a function from $F(X)$ into $P(X)$ satisfying the following condition :

$p \in c(\Phi)$ if and only if $f_i(\Phi)$ c_i-converges to $f_i(p)$ for each $i \in I$

for each $\Phi \in F(X)$. Then we obtain a fuzzy convergence structure c on X that is called the *fuzzy initial convergence structure* induced by the family $\{f_i \mid i \in I\}$ (or $\{c_i \mid i \in I\}$) [7].

2. Fuzzy pseudo-topological structures

A fuzzy convergence space (X, c) is said to be *fuzzy pseudo-topological space* if Φ c-converges to fuzzy point p whenever each ultraprefilter finer than Φ c-converges to p.

Theorem 2.1. Let (X, c) be fuzzy convergence space, and $\rho(c)$ be a function from $F(X)$ to $P(X)$ defined by

$p \in \rho(c)(\Phi)$ if and only if $p \in c(\Psi)$ for each ultraprefilter Ψ finer than Φ

for each $\Phi \in F(X)$. Then $\rho(c)$ is the finest fuzzy pseudo-topological structure coarser than c.

Proof. It is clear that $\rho(c)$ is a fuzzy pseudo-topological sucture coarser than c.

Let c' be fuzzy pseudo-topological structure such that $\rho(c) \leq c' \leq c$ and $p \in \rho(c)(\Phi)$ for each $\Phi \in F(X)$. Then $p \in c(\Psi)$ for each ultraprefilter Ψ finer than Φ. Since $c' \leq c$, we have $p \in c'(\Psi)$. Hence $p \in c'(\Phi)$ and so $c' \leq \rho(c)$. Thus $\rho(c) = c'$.

For each ultraprefilter Ψ on X, by definition of $\rho(c)$, c and $\rho(c)$ have the same ultraprefiler convergence. Thus, by Theorem 2.1, c is fuzzy pseudo-topological structure if and only if $c = \rho(c)$.

Lemma 2.2([7]). Let c_i and c_i' be two fuzzy convergence structures on X_i such that $c_i' \leq c_i$ for each $i \in I$. If c and c' the fuzzy initial convergence structures on X induced by the family $\{f_i \mid f_i : X \to (X_i, c_i)\}$ and $\{f_i \mid f_i : X \to (X_i, c_i')\}$ respectively, then $c' \leq c$.

By Lemma 2.2, the following theorem is easily verified.

Theorem 2.3. Let c and c' be the fuzzy initial convergence structures induced by $\{f_i \mid f_i : X \to (X_i, c_i)\}$ and $\{f_i \mid f_i : X \to (X_i, \rho(c_i))\}$ respectively. Then the following statements hold :

(1) $c' \leq \rho(c)$.

(2) If c_i is fuzzy pseudo-topological for each $i \in I$, then $c' = \rho(c)$.

For each $i \in I$, (X_i, c_i) means fuzzy compact convergence space such that $\alpha_{c_i}(\Phi)$ is one point set for each $\Phi \in F(X_i)$. (X, c), $(\prod X_i, c')$ and $(\prod X_i, c'')$ mean fuzzy initial convergence space induced by $\{f_i \mid f_i : X \to (X_i, c_i)\}$, $\{P_i \mid P_i : X \to (X_i, c_i)\}$ and $\{P_i \mid P_i : X \to (X_i, \rho(c_i))\}$, respectively, where f_i is surjection and P_i is canonical projection. Also, $\prod X_i$, $\prod \rho X_i$ and $\rho \prod X_i$ will denote $(\prod X_i, c')$, $(\prod X_i, c'')$ and $(\prod X_i, \rho(c'))$, respectively.

Theorem 2.4. If (X_i, c_i) is fuzzy pseudo-topological convergence space for each $i \in I$, then initial convergence space (X, c) is also fuzzy pseudo-topological.

Proof. Let Φ be prefilter on X and $p \in c(\Psi)$ for any ultraprefilter Ψ finer than Φ. Then $f_i(\Phi) \subseteq f_i(\Psi)$ and ultraprefilter $f_i(\Psi)$ c_i-converges to $f_i(p)$ for each $i \in I$. Since (X_i, c_i) is fuzzy compact, there exists only one fuzzy point p_i in X_i such that $p_i \in c(f_i(\Psi))$. Since $f_i(\Phi) \subset f_i(\Psi)$ and $\alpha_{c_i}(f_i(\Phi)) = \{f_i(p)\}$, we have $p_i = f_i(p)$. Thus $f_i(\Phi)$ c_i-converges to $f_i(p)$ because (X_i, c_i) is fuzzy pseudo-topological for each $i \in I$. Therefore (X, c) is fuzzy pseudo-topological.

Corollary 2.5. If (X_i, c_i) is fuzzy pseudo-topological convergence space for each $i \in I$, then initial convergence space $(\prod X_i, c')$ is also fuzzy pseudo-topological.

Theorem 2.6. Let (X_i, c_i) be a fuzzy convergence space and let $f_i : X \to (X_i, c_i)$ and $g_i : X \to (X_i, \rho(c_i))$ are surjections defined by $f_i = g_i$ in underlying set for each $i \in I$. If c^* is fuzzy initial convergence structure induced by the family $\{g_i \mid i \in I\}$, then $c^* \leq \rho(c) \leq c$.

Proof. For each $\Phi \in F(X)$, if $p \in c(\Phi)$, then we have

$$g_i(p) = f_i(p) \in c_i(f_i(\Phi)) \subset \rho(c_i)(f_i(\Phi)) = \rho(c_i)(g_i(\Phi)).$$

Since c^* is fuzzy initial convergence stucture induced by $\{g_i \mid i \in I\}$, $p \in c^*(\Phi)$. Hence $c^* \leq c$. By Theorem 2.4, c^* is fuzzy pseudo-topological because $\rho(c_i)$ is fuzzy pseudo-topological. Since $\rho(c)$ is the finest fuzzy pseudo-topological coarser than c, $c^* \leq \rho(c) \leq c$.

Theorem 2.7. If (X_i, c_i) is fuzzy convergence space for each $i \in I$, then $\prod \rho X_i \leq \rho \prod X_i \leq \prod X_i$.

Proof. By definition of fuzzy pseudo-topological, $\rho \prod X_i \leq \prod X_i$. We shall show that $\prod \rho X_i \leq \rho \prod X_i$. Let Φ be a prefilter on $\prod X_i$. If Φ $\rho(c')$- converges to $(p_i)_{i \in I}$ in $\rho \prod X_i$, then Ψ c'-converges to $(p_i)_{i \in I}$ in $\prod X_i$ for all ultraprefilter Ψ finer than Φ. That is, $P_i(\Psi)$ c_i-converges to p_i in X_i for each $i \in I$. Since $\rho(c_i) \leq c_i$, $P_i(\Psi)$ $\rho(c_i)$-converges to p_i in X_i for each $i \in I$. Thus Ψ c'' -converges to $(p_i)_{i \in I}$ in $\prod \rho X_i$. By Corollary 2.5, $\prod \rho X_i$ is a fuzzy pseudo-topological and hence Φ c''-converges to $(p_i)_{i \in I}$ in $\prod \rho X_i$.

A fuzzy convergence space (X, c) is said to be *fuzzy almost pseudo-topological* if $c(\Psi) = \rho(c)(\Psi)$ for all ultraprefilter Ψ on X.

Theorem 2.8. If (X_i, c_i) is fuzzy almost pseudo-topological space for each $i \in I$, then initial convergence space (X, c) is also fuzzy almost pseudo-topological.

Proof. Let c^* be the fuzzy initial convergence structure defined in Theorem 2.6. If $p \in c^*(\Phi)$, then for each $i \in I$, $g_i(p) \in \rho(c_i)(g_i(\Phi))$. Since (X_i, c_i) is fuzzy almost pseudo-topological, $f_i(p) = g_i(p) \in \rho(c_i)(f_i(\Psi)) = c_i(f_i(\Psi))$ for all ultra-prefilter Ψ on X. Thus $p \in c(\Psi)$ and hence $c^*(\Psi) \subset c(\Psi) \subset \rho(c)(\Psi)$. By Theorem 2.6, $c(\Psi) = c^*(\Psi) = \rho(c)(\Psi)$. Therefore, (X, c) is fuzzy almost pseudo-topological.

Corollary 2.9. If (X_i, c_i) is a fuzzy almost pseudo-topological for each $i \in I$, then initial convergence space $(\prod X_i, c')$ is a fuzzy almost pseudo-topological.

$(X_i, c_i)_{i \in I}$ of fuzzy convergence space (X_i, c_i) is said to be a *fuzzy pseudo-topologically coherent* if $\rho \prod X_i = \prod \rho X_i$.

Theorem 2.10. Let (X_i, c_i) be fuzzy convergence space for each $i \in I$. Then (X_i, c_i) is a fuzzy almost pseudo-topological for each $i \in I$ if and only if $(X_i, c_i)_{i \in I}$ is a fuzzy pseudo-topologically coherent.

Proof. Suppose that $\prod \rho X_i < \rho \prod X_i$. Then there exists a prefilter Φ on $\prod X_i$ such that Φ converges to $(p_i)_{i \in I}$ in $\prod \rho X_i$ and does not in $\prod X_i$. By definition of $\rho \prod X_i$, there exists a ultraprefilter Ψ such that Ψ does not c'-converges to $(p_i)_{i \in I}$ in $\prod X_i$ and $\Phi \subset \Psi$. But Ψ converges to $(p_i)_{i \in I}$ in $\prod \rho X_i$ because Φ

converges to $(p_i)_{i\in I}$ in $\prod \rho X_i$ and $\Psi \supset \Phi$. Thus $P_i(\Psi)$ $\rho(c_i)$-converges to p_i in ρX_i for each $i\in I$, so that $P_i(\Psi)$ c_i-converges to p_i in X_i for each $i\in I$ by definition of fuzzy almost pseudo-topological convergence space. Hence Ψ c'-converges to $(p_i)_{i\in I}$ in $\prod X_i$. This contradicts to the fact that a ultraprefilter Ψ does not c'-converge to $(p_i)_{i\in I}$ in $\prod X_i$. Thus $(X_i, c_i)_{i\in I}$ is fuzzy pseudo-topologically coherent.

Conversely, suppose that $(X_i, c_i)_{i\in I}$ is fuzzy pseudo-topologically coherent, and let Φ_i be arbitrary ultraprefilter on X_i. If Φ_i c_i-converges to p_i in X_i for each $i\in I$, then Φ_i $\rho(c_i)$-converges to p_i in ρX_i for each $i\in I$. On the other hand, if Φ_i $\rho(c_i)$-converges to p_i in ρX_i for each $i\in I$, then $\prod\Phi_i$ converges to $(p_i)_{i\in I}$ in $\prod\rho X_i$. Since $\rho\prod X_i = \prod\rho X_i$, $\prod\Phi_i$ $\rho(c')$ -converges to $(p_i)_{i\in I}$ in $\rho\prod X_i$ and so Φ c'-converges to $(p_i)_{i\in I}$ in $\prod X_i$ for all ultraprefilter Φ finer than $\prod\Phi_i$, thus $P_i(\Phi)$ c_i-converges to p_i in X_i for each $i\in I$. Since $P_i(\Phi)$ and Φ_i are ultraprefilters on X_i and $P_i(\Phi) \supset P_i(\prod\Phi_i) = \Phi_i$, $P_i(\Phi) = \Phi_i$ c_i-converges to p_i in X_i for each $i\in I$. Thus X_i and ρX_i have the same ultraprefilter convergence and hence, for each $i\in I$, X_i is a fuzzy almost pseudo-topological.

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