A Survey on Proofs of the Tychonoff Theorem

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1. Introduction and Preliminaries

The Tychonoff theorem is one of the most important theorems in general topology. It plays a central role in the development of a wealth of theorems within topology and applications of topology to other fields; the construction of the Stone-Cech compactification of any Tychonoff space, Ascoli's theorem on compactness of function spaces, the proof of compactness of the maximal ideal space of a Banach algebra, the study of Cantor set, etc... In this note, in order to reflect on the Tychonoff theorem, we will introduce several proofs of the Tychonoff theorem which use as basic tools the Axiom of choice, net, filter, and subbase.

We introduce some definitions and theorems which will be used throughout this note. Let X_λ $(\lambda \in \Lambda)$ be a set, then the product set is $X = \prod_{\lambda \in \Lambda} X_\lambda = \{x | x : \Lambda \to \bigcup_{\lambda \in \Lambda} X_\lambda,\ x(\lambda) \in X_\lambda, \lambda \in \Lambda\}$. The Tychonoff theorem says that X_λ $(\lambda \in \Lambda)$ is compact iff $X = \prod_{\lambda \in \Lambda} X_\lambda$ is compact.

The following statements are equivalent .

1. The Axiom of choice :To any nonempty set $\mathcal{T}$ whose elements are nonempty sets, there exists a function called a choice function

$$f : \mathcal{T} \to \bigcup_{A \in \mathcal{T}} A \quad \text{such that} \quad f(A) \in A \quad \text{for all } A \in \mathcal{T}.$$

2. X_λ $(\lambda \in \Lambda)$ is nonempty set, then $X = \prod_{\lambda \in \Lambda} X_\lambda$ is nonempty set

3. Zorn's lemma : Let P be a partially ordered set in which every chain has an upper bound. Then P has a maximal element.

Definition 1.1. A class A of subset of X has the finite intersection property iff the intersection of any finite subclass from A is nonempty.

Definition 1.2. A set Λ is directed set iff there is a relation $\leq$ on Λ satisfying:

(1) $\lambda \leq \lambda$, for each $\lambda \in \Lambda$, (2) if $\lambda_1 \leq \lambda_2$, $\lambda_2 \leq \lambda_3$, then $\lambda_1 \leq \lambda_3$,

(3) if $\lambda_1, \lambda_2 \in \Lambda$, then there is some $\lambda_3 \in \Lambda$ with $\lambda_1 \leq \lambda_3$, $\lambda_2 \leq \lambda_3$.

Definition 1.3. A net (x_λ) in a set X is a function $f: \Lambda \to X$, where Λ is some directed set. The point $f(\lambda)$ is usually denoted by x_λ, and we often speak of " the net (x_λ) ".

Definition 1.4. A net (x_λ) in a set X is an ultranet iff for each subset E of X, there exists $\lambda_0 \in \Lambda$ such that either $x_\lambda \in E$ or $x_\lambda \in X - E$ for all $\lambda \geq \lambda_0$.

Definition 1.5. Let (x_λ) be a net in a topological space X. Then (x_λ) converges to x (written $(x_\lambda) \to x$) provided for each nbd U of x, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$.

Lemma 1.6. If (x_λ) is an ultranet in X and f is a map: $X \to Y$, then $(f(x_\lambda))$ is an ultranet in Y.

Definition 1.7. A filter F on a set X is a nonempty collection of nonempty subsets of X with the properties:

If $F_1, F_2 \in F$ then $F_1 \cap F_2 \in F$, if $F_1 \in F$, $F_1 \subset F_2$ then $F_2 \in F$.

A subcollection F_0 of F is a filterbase for F iff each element of F containsome element of F_0, that is, iff $F = \{F_1 \subset X \mid F_\lambda \subset F_1$, for some $F_\lambda \in F_o\}$.

Definition 1.8. A nonempty collection C of nonempty subsets of X is a filterbase for some filter on X iff if $C_1, C_2 \in C$ then $C_3 \subset C_1 \cap C_2$ for some $C_3 \in C$, in which case the filter G generated by C consists of all supersets of elements of C, namely $G = \{G_\lambda \subset X \mid C_\lambda \subset G_\lambda,$ for some $C_\lambda \in C\}$.

Definition 1.9. If F is a filter on X and f maps X into Y, then $f(F) = \{G_\lambda \subset Y \mid f(F_1) \subset G_\lambda,$ for some $f(F_1) \in C\}$ is the filter on Y having for a filterbase $C = \{f(F_1) \mid F_1 \in F\}$.

Definition 1.10. A filter F on X is an ultrafilter iff there is no strictly finer filter G than F, that is, there does not exist G such that $F \subset G$.

Lemma 1.11. If F is an ultrafilter on X and f maps X into Y, then $f(F) = \{G_\lambda \subset Y \mid f(F_1) \subset G_\lambda,$ for some $f(F_1) \in C\}$ is an ultrafilter on Y having for a filterbase $C = \{f(F_1) \mid F_1 \in F\}$.

Definition 1.12. A filter F on a topological space X is said to converge to x (written by $F \to x$) if the nbd system U_x at x is contained in F.

Definition 1.13. If (x_λ) is a net in X,
the filter $G = \{G_\lambda \subset X \mid B_{\lambda_0} \subset G_\lambda$ for some $B_{\lambda_0} \in C\}$ generated by the filterbase $C = \{B_{\lambda_0} \mid \lambda_0 \in \Lambda\}$, $B_{\lambda_0} = \{x_\lambda \mid \lambda \geq \lambda_0\}$, is called the filter G generated by (x_λ).

Definition 1.14. If F is a filter on X, let $\Lambda_F = \{(x, F_\alpha) \mid x \in F_\alpha \in F\}$. Then Λ_F is directed by the relation $(x_1, F_{\alpha_1}) \leq (x_2, F_{\alpha_2})$ iff $F_{\alpha_2} \subset F_{\alpha_1}$, so the map $P : \Lambda_F \to X$ defined by $P(x, F_\alpha) = x$ is a net in X. It is called the net based on F.

Definition 1.15. Let (X, T) be a topological space. A class S of open subsets of X is a subbase for the topology T on X iff the finite intersections of members of S form a base for T.

The following statements are equivalent and play a crucial role in the proof;
(1) X is compact,
(2)Each open cover of X has a finite subcover ,
(3)Each family F of closed subsets of X with the finite intersection property has a nonempty intersection,
(4) Each ultranet in X converges,
(5) Each ultrafilter in X converges,
(6) There is a subbase S for X that each subfamily SL of S, such that no finite subfamily of SL covers X, fails to cover X,
(7) Each family B of open subsets of X, such that no finite subfamily of B covers X, fails to cover X.

2. Proof of the Tychonoff theorem by Zorn's Lemma.

Lemma 2.1. *Let J be a class of subsets of a set X with the finite intersection property. Consider the collection F of all superclasses of J which have the finite intersection property. Then F, ordered by class inclusion, contains a maximal element* M [4,p.174].

Proof. Let $T=\{ B_\lambda | \lambda \in \Lambda \}$ be a chain of F and let $B=\bigcup_{\lambda\in\Lambda} B_\lambda$.

Claim : $B\in F$,i.e $J\subset B$, B has the finite intersection property. Since $B_\lambda \in$P ($\lambda\in\Lambda$), J $\subset$ P, then $J\subset B_\lambda$. Since $B_\lambda \subset B$, then $J\subset B$. Let

$\{A_1, A_2, \ldots, A_n\} \subset B$. Since $B = \bigcup_{\lambda \in \Lambda} B_\lambda$, there exists $B_{\lambda_1}, \ldots, B_{\lambda_n} \in T$ such that $A_1 \in B_{\lambda_1}, \ldots, A_n \in B_{\lambda_n}$. Since T is a chain of F, there exists B_{λ_k}, $1 \leq k \leq n$ such that $B_{\lambda_1} \subset B_{\lambda_k}$, $B_{\lambda_2} \subset B_{\lambda_k}$, ..., $B_{\lambda_n} \subset B_{\lambda_k}$.Hence $\{A_1, A_2, \ldots, A_n\} \subset B_{\lambda_k}$. Since B_{λ_k} has the finite intersection property, then $\bigcap_{i=1}^{n} A_i \neq \emptyset$ i.e B has the finite intersection property. For all $B_\lambda \in T$, we have $B_\lambda \subset \bigcup B_\lambda = B$ i.e B is an upper bound for T. By Zorn's lemma, F contains a maximal element.

Lemma 2.2. *The maximal element M in Lemma 2.1 posses the following properties [4,p.175];*

(1) if $\{M_1, M_2, \ldots, M_n\} \subset M$, *then* $M_1 \cap M_2 \ldots \cap M_n \in M$,

(2) if $A \cap M_1 \neq \emptyset$, for *every* $M_1 \in M$, *then* $A \in M$.

The Tychonoff theorem says that X_λ $(\lambda \in \Lambda)$ is compact iff $X = \prod_{\lambda \in \Lambda} X_\lambda$ is compact[4,p.175].

Proof. (<===) Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be compact. Since the projection maps $P_\lambda : X \to X_\lambda$ $(\lambda \in \Lambda)$ are allcontinuous, $P_\lambda[X] = X_\lambda$ is compact.

(===>) Let X_λ $(\lambda \in \Lambda)$ be compact. Let $J = \{F_k \mid k \in K\}$ be a class of closed subsets of a set $X = \prod_{\lambda \in \Lambda} X_\lambda$ with the finite intersection property. Then, we will show that $\bigcap J = \bigcap\{F_k \mid k \in K\} \neq \emptyset$. By Lemma 2.1, $M = \{M_\eta \mid \eta \in H\}$ be a maximal superclass of J with the finite intersection property. For each projection maps $P_\lambda : X \to X_\lambda$ $(\lambda \in \Lambda)$, $\{ \overline{P_\lambda[M_\eta]} \mid \eta \in H \}$ is a class of closed subsets of X_λ with the finite intersection property.

[$\because$ for $\{M_{\eta_1}, M_{\eta_2}, \cdots, \} \subset M$ then $\emptyset \neq \bigcap_{i=1}^{n} M_{\eta_i} \in M$. Since

$P_\lambda[M_{\eta_i}] \subseteq \overline{P_\lambda[M_{\eta_i}]}$, then $\emptyset \neq P_\lambda\left(\bigcap_{i=1}^{n} M_{\eta_i}\right) \subseteq \bigcap_{i=1}^{n} P_\lambda[M_{\eta_i}] \subseteq \bigcap_{i=1}^{n} \overline{P_\lambda[M_{\eta_i}]}$]

Since X_λ $(\lambda \in \Lambda)$ is compact, $\bigcap \{ \overline{P_\lambda[M_\eta]} \mid \eta \in H \} \neq \emptyset$, $(\lambda \in \Lambda)$. Taking $x_\lambda \in \bigcap \{ \overline{P_\lambda[M_\eta]} \mid \eta \in H \} \neq \emptyset$ $(\lambda \in \Lambda)$, then we have for every $\eta \in H$, $x_\lambda \in \overline{P_\lambda[M_\eta]}$ i.e for every open set G_λ of x_λ,

$$G_\lambda \cap P_\lambda[M_\eta] \neq \emptyset \quad (\eta \in H) \tag{2.1}$$

Here, let $p = \langle x_\lambda \mid \lambda \in \Lambda \rangle$, then $p \in X = \prod_{\lambda \in \Lambda} X_\lambda$. Let $p \in B$, where B is a member of the defining base for $X = \prod_{\lambda \in \Lambda} X_\lambda$; $(j = 1, 2, 3, \cdots, n)$

$p \in B = P_{\lambda_1}^{-1}[G_{\lambda_1}] \cap \cdots \cap P_{\lambda_n}^{-1}[G_{\lambda_n}]$, where G_{λ_j} is open set on X_{λ_j}.

Since $P_{\lambda_j}(p) = x_{\lambda_j} \in G_{\lambda_j}$, by(2.1), $G_{\lambda_j} \cap P_{\lambda_j}[M_\eta] \neq \emptyset$ $(\eta \in H), (j = 1, 2, 3, \ldots, n)$.

Hence $P_{\lambda_j}^{-1}[G_{\lambda_j}] \cap M_\eta \neq \emptyset$. By Lemma 2.2.2, $P_{\lambda_j}^{-1}[G_{\lambda_j}] \in M$.By Lemma2.2.1, $B = P_{\lambda_1}^{-1}[G_{\lambda_1}] \cap \cdots\cdots \cap P_{\lambda_n}^{-1}[G_{\lambda_n}] \in M$. Since M has the finite intersection property, for every $M_\eta \in M$,

$$B \cap M_\eta = P_{\lambda_1}^{-1}[G_{\lambda_1}] \cap \cdots\cdots \cap P_{\lambda_n}^{-1}[G_{\lambda_n}] \cap M_\eta \neq \emptyset$$

i.e $p \in \overline{M_\eta}$ $(\eta \in H)$. Since $J \subset M$, $J = \{ F_k \mid k \in K \}$ is a class of closed subsets of $X = \prod_{\lambda \in \Lambda} X_\lambda$, for every $F_k \in J$, $p \in \overline{F_k} = F_k$ $(k \in K)$. Hence $p \in \bigcap J = \bigcap \{F_k \mid k \in K\} \neq \emptyset$ i.e $X = \prod_{\lambda \in \Lambda} X_\lambda$ is compact.

3. Proof of the Tychonoff theorem by Net.

Lemma 3.1 $(P_\alpha(x_\lambda)) \to P_\alpha(x)$ *in* X_α $(\alpha\in A)$, *then a net* $(x_\lambda) \to x$ *in* $X = \prod_{\alpha\in A} X_\alpha$ *[5,p.76].*

Proof. Let $U_x=\{ U\subset X \mid U$ is a nbd of $x \}$ be the nbd system at x in X, where $U=\bigcap_{i=1}^{n} P_{\alpha_i}{}^{-1}[U_{\alpha_i}]$, U_{α_i} is a nbd of $P_{\alpha_i}(x)$ in X_{α_i}.Since $(P_\alpha(x_\lambda)) \to P_\alpha(x)$ in X_α, then for each U_{α_i} of $P_{\alpha_i}(x_\lambda)$ in X_{α_i}, there exists $\lambda_i\in\Lambda$ such that $P_{\alpha_i}(x)\in U_{\alpha_i}$ for all $\lambda\geq\lambda_i$. Here let $\lambda_0=MAX\{\lambda_i, 1\leq i\leq n\}$, then we have $P_{\alpha_i}(x) \in U_{\alpha_i}$ for all $\lambda\geq\lambda_0$. Then $x_\lambda \in U=\bigcap_{i=1}^{n} P_{\alpha_i}{}^{-1}[U_{\alpha_i}]$ for all $\lambda\geq\lambda_0$. Hence a net $(x_\lambda) \to x$ in X.

The Tychonoff theorem says that X_α $(\alpha\in A)$ is compact iff $X=\prod_{\alpha\in A} X_\alpha$ is compact[5,p.120].

Proof.(===>) Let (x_λ) be an ultranet in $X=\prod_{\alpha\in A} X_\alpha$. By Lemma 1.6, $(P_\alpha(x_\lambda))$ is an ultranet in X_α. Since X_α is compact, then $(P_\alpha(x_\lambda))$ converges in X_α. By Lemma 3.1, (x_λ) converges in $X=\prod_{\alpha\in A} X_\alpha$. Hence $X=\prod_{\alpha\in A} X_\alpha$ is compact.

4. Proof of the Tychonoff theorem by Filter.

Lemma 4.1 $P_\lambda(F) \rightarrow P_\lambda(x)$ *in* X_λ $(\lambda \in \Lambda)$, *then a filter* $F \rightarrow x$ *in* $X = \prod_{\lambda \in \Lambda} X_\lambda$ *[1,p.217].*

Proof. Let $U_x = \{ U \subset X \mid U \text{ is a nbd of } x \}$ be the nbd system at x in X, where $U = \bigcap_{i=1}^{n} P_{\lambda_i}^{-1}[U_i]$, U_i is a nbd of $P_{\lambda_i}(x)$ in X_{λ_i}.
Since $P_\lambda(F) \rightarrow P_\lambda(x)$ in X_λ, then $U_i \in P_{\lambda_i}[F]$. Then $P_{\lambda_i}[F_i] \subset U_i$ for some $F_i \in F$. then we have $F_i \subset P_{\lambda_i}^{-1}[U_i]$. Since F is a filter, $\bigcap_{i=1}^{n} F_i \in F$, $\bigcap_{i=1}^{n} F_i \subset U = \bigcap_{i=1}^{n} P_{\lambda_i}^{-1}[U_i]$, so $U = \bigcap_{i=1}^{n} P_{\lambda_i}^{-1}[U_i] \in F$. Thus the nbd system at x $U_x \subset F$. Hence a filter $F \rightarrow x$ in $X = \prod_{\lambda \in \Lambda} X_\lambda$.

The Tychonoff theorem says that X_λ $(\lambda \in \Lambda)$ is compact iff $X = \prod_{\lambda \in \Lambda} X_\lambda$ is compact[1,p.224].

Proof. (===>) Let F be an ultrafilter on X. By lemma 1.11., $P_\lambda(F)$ is an ultrafilter on X_λ. Since X_λ is compact, then $P_\lambda(F)$ converges in X_λ. By lemma 4.1., F converges in X. Hence X is compact.

Remarks. A sequence is not sufficient to explain a convergence in a set. In order to explain to the convergence in a set, the conception of a net (x_λ) is generated. A filter is generated in the process of deeply investigating a tail $B_{\lambda_0} = \{x_\lambda \mid \lambda \geq \lambda_0\}$ of U.

The following properties are obtained by Definition(1.1.3) (1.1.4) [5,p81];

(1) A filter F *converges to* x *in* X *iff the net* ($P(x,F_\alpha)$) *based on* F *converges to* x,

(2) A net (x_λ) *converges to* x *in* X *iff the filter* G *generated by* (x_λ) *converges to* x.

Proof (1). (===>) Let $F \to x$. Let $U_x = \{ U \subset X \mid U \text{ is a nbd of } x \}$ be the nbd system at x in X. Since $F \to x$, then $U_x \subset F$. Hence $U \in F$. Pick $p \in U$. Then $(p, U) \in \Lambda_F$ and if $(q, F_\alpha) \geq (p, U)$, then $F_\alpha \subset U$, $P(q, F_\alpha) = q \in F_\alpha \subset U$. Hence the net ($P(x,F_\alpha)$) based on F converges to x.

(<===) Let the net ($P(x,F_\alpha)$) based on F converges to x. Let $U_x = \{ U \subset X \mid U \text{ is a nbd of } x \}$ be the nbd system at xin X. Since the net ($P(x,F_\alpha)$) based on F converges to x, for all $U \in U_x$, there exists some (p_0, F_{α_0}) such that $P(p, F_\alpha) = p \in U$ for all $(p, F_\alpha) \geq (p_0, F_{\alpha_0})$, i.e for all $p \in F_\alpha$, then $p \in U$. Hence $F_\alpha \subset U$. Since F is a filter, then $U \in F$. Hence $U_x \subset F$, i.e $F \to x$.

5.Proof of the Tychonoff theorem by Subbase.

The Tychonoff theorem says that X_λ $(\lambda \in \Lambda)$ is compact iff $X = \prod_{\lambda \in \Lambda} X_\lambda$ is compact[2,p.143].

Proof. (===>) Let $S = \bigcup_{\lambda \in \Lambda} \{ P_\lambda^{-1}(U) \mid U \text{ is open in } X_\lambda \}$ be a subbase for $X = \prod_{\lambda \in \Lambda} X_\lambda$, such that no finite subfamily of S covers $X = \prod_{\lambda \in \Lambda} X_\lambda$. Then we will show that S fails to cover $X=\prod_{\lambda\in\Lambda} X_\lambda$. for each $\lambda\in\Lambda$,let

$$B_\lambda=\{\, U\subset X_\lambda \mid U \text{ is open in } X_\lambda \text{ such that } P_\lambda^{-1}(U)\in S \,\}.$$

Since $\bigcup_{i=1}^{n} P_\lambda^{-1}(U_i)$ fails to cover $X=\prod_{\lambda\in\Lambda} X_\lambda$, $\bigcup_{i=1}^{n} U_i$ fails to cover X_λ ,that is, no finite subfamily of B_λ cover X_λ. Since X_λ is compact, B_λ fails to cover X_λ. Then, there is a point x_λ such that $x_\lambda\in(X_\lambda-U)$ for each Uin B_λ. Then point x whose $\lambda-$th coordinate is x_λ belongs to no member of S and consequently S fails to cover $X=\prod_{\lambda\in\Lambda} X_\lambda$. Hence $X=\prod_{\lambda\in\Lambda} X_\lambda$ is compact.

6.The Tychonoff product theorem implies the Axiom of choice.

A sketch of the proof of J.L.Kelley[2] is as follows. He assuredly demonstrate the following statement of the Axiom of choice :

$$X_\lambda \neq \emptyset \quad (\lambda\in\Lambda), \quad \text{then} \quad X=\prod_{\lambda\in\Lambda} X_\lambda \neq \emptyset.$$

Step 1. He begin by adjoining a single point, say A, to each of the set X_λ: Let $Y_\lambda=X_\lambda\cup\{A\}$.

Step 2. He assign a topology for Y_λ by defining the cofinite topolgy T_λ on Y_λ, then $T_\lambda=\{G_\lambda \mid G_\lambda^c \text{ is a finite subset of } Y_\lambda\}\cup\{\emptyset\}$. Then Y_λ is compact and the product space $Y=\prod_{\lambda\in\Lambda} Y_\lambda$ is compact by the Tychonoff theorem.

Step 3. For each $\lambda\in\Lambda$, let Z_λ be the subset $P_\lambda^{-1}(X_\lambda)=X_\lambda\times\prod_{\lambda\neq\eta} Y_\eta$ of Y.

Step 4. He assumed that X_λ is closed in Y_λ and Z_λ is closed in Y.

Step 5. For any finite subset B of Λ, the intersection $\emptyset\neq\bigcap_{\lambda\in B} Z_\lambda =$

$\bigcap_{\lambda \in B} P_\lambda^{-1}(X_\lambda)$, for, since each $X_\lambda \neq \emptyset$, he may by the finite Axiom of choice choose $x_\lambda \in X_\lambda$ for $\lambda \in B$, and set $x_\lambda = A$ for $\lambda \in \Lambda - B$.

Step 6. The family of all sets of the form Z_λ is a family of closed subsets of Y with the property that the intersection of any finite subfamily is nonempty. Since Y is compact, we have $\emptyset \neq \bigcap_{\lambda \in \Lambda} Z_\lambda = \bigcap_{\lambda \in \Lambda} P_\lambda^{-1}(X_\lambda)$. Bu $\bigcap_{\lambda \in \Lambda} P_\lambda^{-1}(X_\lambda)$ is precisely $X = \prod_{\lambda \in \Lambda} X_\lambda$, and the Axiom of choice is proved.

But, Step 4 is not true. We assert that X_λ is not closed but open in Y_λ and Z_λ is not closed but open in Y. First, we show that X_λ is not closed in Y_λ. Indeed, if X_λ were closed in Y_λ, then X_λ^c is open in Y_λ, i.e $X_\lambda^c = \{A\}$ is open in Y_λ. Hence $\{A\}^c = X_\lambda$ is a finite set. But this is not true in the case that X_λ is an infinite set.
On the other hand, by the definition of the cofinite topology T_λ for Y_λ, $\{A\}^c = X_\lambda$ is clearly open in Y_λ. Now we show that Z_λ is not closed in Y. Indeed, if Z_λ were closed in Y, then $\{Z_\lambda\}^c = \{A\} \times \prod_{\eta \neq \lambda} Y_\eta$ $(\eta \in \Lambda)$ is open Y. Hence there exist the finite nonempty open subsets O_{α_i} in Y_{α_i} $(\alpha_i, \alpha \in \Lambda)$ $(i = 1, 2, \cdots, n)$ such that

$$O_{\alpha_1} \times O_{\alpha_2} \times \cdots \times O_{\alpha_n} \times \prod_{\alpha \neq \alpha_i} Y_\alpha \subseteq \{A\} \times \prod_{\eta \neq \lambda} Y_\eta.$$

Then, we have two possibilities;
Case1: $\lambda = \alpha_i$, for some i.
Then, we have $\emptyset \neq O_\lambda \subseteq \{A\}$. Since $\{A\}$ is a singleton set, we have $O_\lambda = \{A\}$. By the fact that X_λ is not closed but open in Y_λ, $X_\lambda^c = \{A\}$ is closed but not open in Y_λ. Hence O_λ is closed but not open in Y_λ. This contradicts the fact that O_λ is open in Y_λ.

Case2: $\lambda=\alpha$, for some $\alpha\in\Lambda$.

Then, we have $Y_\lambda\subseteq\{A\}$. Since $\{A\}$ is a singleton set, we have $Y_\lambda=\{A\}$. Since $Y_\lambda=X_\lambda\cup\{A\}$, we have $X_\lambda=\emptyset$. This contradicts $X_\lambda\neq\emptyset$. This completes the proof. Now we show that Z_λ is open in Y. Indeed, we know that X_λ is open in Y_λ. Since the projection maps P_λ's are all continuous, we have $Z_\lambda= P_\lambda^{-1}(X_\lambda)= X_\lambda\times\prod_{\eta\neq\lambda}Y_\eta$ is open in Y. Finally, we are in the position to give a correct proof of the theorem. We begin by adjoining a single point, say A, to each of the set X_λ: Let $Y_\lambda=X_\lambda\cup\{A\}$. We assign a topology for Y_λ by defining $\mathfrak{I}_\lambda=\{\emptyset,\{A\},X_\lambda,Y_\lambda\}$. Since $\mathfrak{I}_\lambda$ is a finite set, each open cover of Y_λ obviously has a finite subcover. Thus Y_λ is compact and the product space $Y=\prod_{\lambda\in\Lambda}Y_\lambda$ is compact by the Tychonoff theorem. Note that X_λ is closed in Y_λ by the definition of the topology $\mathfrak{I}_\lambda$. For each $\lambda\in\Lambda$, let Z_λ be the subset $P_\lambda^{-1}(X_\lambda)=X_\lambda\times\prod_{\lambda\neq\eta}Y_\eta$ $(\eta\in\Lambda)$ of Y.

Then $Z_\lambda= P_\lambda^{-1}(X_\lambda)$ is closed in Y. Indeed, we know that X_λ is closed in Y_λ. Since the projection maps P_λ's are all continuous, we have $Z_\lambda= P_\lambda^{-1}(X_\lambda)= X_\lambda\times\prod_{\eta\neq\lambda}Y_\eta$ is closed in Y. Moreover, for any finite subset B of Λ, the intersection $\emptyset\neq\bigcap_{\lambda\in B}Z_\lambda=\bigcap_{\lambda\in B}P_\lambda^{-1}(X_\lambda)$, for, since each $X_\lambda\neq\emptyset$, we may by the finite Axiom of choice choose $x_\lambda\in X_\lambda$ for $\lambda\in B$, and set $x_\lambda=A$ for $\lambda\in\Lambda-B$. Consequently, the family of all sets of the form Z_λ is a family of closed subsets of Y with the property that the intersection of any finite subfamily is nonempty. Since Y is compact, we have $\emptyset\neq\bigcap_{\lambda\in\Lambda}Z_\lambda=\bigcap_{\lambda\in\Lambda}P_\lambda^{-1}(X_\lambda)$. But, $\bigcap_{\lambda\in\Lambda}P_\lambda^{-1}(X_\lambda)$ is precisely $X=\prod_{\lambda\in\Lambda}X_\lambda$, and the Axiom of choice is proved.

Remark. J.L.Kelley's proof is incorrect because of assignment of the cofinite topology $\mathcal{T}_\lambda$ for Y_λ. But we can prove the desired result just using his argument, only if we assign a new topology for Y_λ with the properties:

X_λ is closed in Y_λ, Y_λ is compact, and Z_λ is closed in Y.

Simply we find another example for which is satisfying,

that is, $\mathfrak{T}_\lambda = \{\emptyset, \{A\}, Y_\lambda\}$.