A Remark on Implicit Vector Variational Inequality

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Abstract

In this paper, we study the existence of solutions of implicit vector variational inequalities for noncompact valued multifunctions under generalized pseudomonotonicity assumptions and the Hausdorff topological vector space setting.

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1. Introduction

A vector variational inequality (shortly, VVI) in a finite dimensional Euclidean space was first introduced by Giannessi [6]. Since then, many authors have studied existence theorems for generalized versions of VVI ([1, 2, 4, 5, 7-20, 22-24]. In particular, Lee and Kum [18] poved some existence theorems for solutions of implicit vector variational inequalities compact valued multifunctions under generalized weak pseudomonotonicity assumptions and the Hausdorff topological vector space setting.

In this paper, following the approaches of Konnov and Yao [9], we investigate the existence of solutions of implicit vector variational inequalities for noncompact valued multifunctions under generalized pseudomonotonicity assumptions and the Hausdorff topological vector space setting.

2. Preliminaries

Let E be a Hausdorff topological vector space, X a nonempty convex subset of E, F another topological vector space and $C: X \to 2^F$ a multifunction such that for each $x \in X$, Cx is a convex cone in F with $int\, Cx \neq \emptyset$ and $Cx \neq F$. Let $L(E, F)$ be the space of all continuous linear mappings from E to F, $\psi: L(E, F) \times X \times X \to F$ a function and $T: X \to 2^{L(E,F)}$ a multifunction.

In this paper, we consider the following implicit vector variational inequality $(IVVI)$ for multifunctions; $s \in T\bar{x}$

$(IVVI)$ Find $\bar{x} \in X$ such that for each $y \in X$, there exists such that

$$\psi(s, \bar{x}, y) \notin -int\, C\bar{x}.$$

We consider now the following special cases of $(IVVI)$. For any $s \in L(E, F)$ and $x \in E$, $\langle s, x \rangle$ denotes the evaluation of s at x.

(I) If $A: L(E, F) \to L(E, F)$ is a continuous mapping and

$\psi(s,x,y)=\langle As, \eta(y,x)\rangle$, then $(IVVI)$ is equivalent to finding $\bar{x}\in X$ such that for each $y\in X$, there exists $s\in T\bar{x}$ such that $\langle As, \eta(y,\bar{x})\rangle \notin -int\,C\bar{x}$, which is the generalized vector variational-like inequality for multifunctions investigated by Ansari [2].

(II) If $\psi(s,x,y)=\langle s, \eta(y,x)\rangle$, then $(IVVI)$ reduces to the problem of finding $\bar{x}\in X$ such that for each $y\in X$, there exists $s\in T\bar{x}$ such that $\langle s, \eta(y,\bar{x})\rangle \notin -int\,C\bar{x}$, which is the vector variational-like inequality for multifunctions studied by Ansari [1] and Lee *et. al.* [11].

(III) If $\psi(s,x,y)=\langle s, y-x\rangle$, then $(IVVI)$ becomes the problem of finding $\bar{x}\in X$ such that for each $y\in X$, there exists $s\in T\bar{x}$ such that $\langle s, y-\bar{x}\rangle \notin -int\,C\bar{x}$, which is the vector variational inequality for multifunctions investigated by Lee *et. al.* [12], Lin *et. al.* [20] and Konnov *et. al.* [9].

(IV) If $\psi(s,x,y)=\langle s, y-x\rangle$ and T is a single-valued map, then $(IVVI)$ is equivalent to finding $\bar{x}\in X$ such that for each $y\in X$, $\langle T\bar{x}, y-\bar{x}\rangle \notin -int\,C\bar{x}$, which is the vector variational inequality for vector-valued functions investigated by Chen [4], Lai *et. al.* [10] and Yu *et. al.* [24].

Now we give the generalized pseudomonotonicity concepts and the generalized hemicontinuity concepts on the multifunction T.

T is said to be

(1) generalized C-pseudomonotone w.r.t. ψ if for any $x, y\in X$, $\exists\, s\in Tx$ such that $\psi(s,x,y)\notin -intCx$ implies $\forall t\in Ty$ $-\psi(t,y,x)\notin -intCx$.

(2) generalized weakly C-pseudomonotone w.r.t. ψ if for any $x, y\in X$, $\exists\, s\in Tx$ such that $\psi(s,x,y)\notin -intCx$ implies $\exists\, t\in Ty$ such that $-\psi(t,y,x)\notin -intCx$.

(3) generalized hemicontinuous w.r.t. ψ if for any $x, y\in X$ and $\alpha\in[0,1]$, the multifunction $\alpha\mapsto\psi(T(x+\alpha(y-x)), x+\alpha(y-x), y)$ is

upper semicontinuous at 0+, where
$\psi(T(x+\alpha(y-x)), x+\alpha(y-x), y) = \{\psi(t, x+\alpha(y-x), y) : t \in T(x+\alpha(y-x))\}$.

Let $\eta : X \times X \to E$ be a function. Then T is said to be

(1)′ generalized C-pseudomonotone w.r.t. η if for any $x, y \in X$ $\exists\, s \in Tx$ such that $\langle s, \eta(y, x)\rangle \notin -intCx$ implies $\forall\, t \in Ty$ $-\langle t, \eta(x, y)\rangle \notin -intCx$.

(2)′ generalized weakly C-pseudomonotone w.r.t. η if for any $x, y \in X$ $\exists\, s \in Tx$ such that $\langle s, \eta(y, x)\rangle \notin -intCx$ implies $\exists\, t \in Ty$ such that $-\langle t, \eta(x, y)\rangle \notin -intCx$.

(3)′ generalized hemicontinuous w.r.t. η if for any $x, y \in X$ and $\alpha \in [0, 1]$, the multifunction $\alpha \mapsto \langle T(x+\alpha(y-x)), \eta(y, x+\alpha(y-x))\rangle$ is upper semicontinuous at 0+, where
$\langle T(x+\alpha(y-x)), \eta(y, x+\alpha(y-x))\rangle = \{\langle s, \eta(y, x+\alpha(y-x))\rangle : s \in T(x+\alpha(y-x))\}$.

We can easily obtain the following lemma.

Lemma 2.1. Let E, X, F, C, η, ψ, and T be the same as in the above. Then we have

(1) T is generalized C-pseudomonotone w.r.t. η $\Rightarrow$ T is generalized C-pseudomonotone w.r.t. some ψ.

(2) T is generalized weakly C-pseudomonotone w.r.t. η $\Rightarrow$ T is generalized weakly C-pseudomonotone w.r.t. some ψ.

(3) T is generalized C-pseudomonotone w.r.t. ψ $\Rightarrow$ T is generalized weakly C-pseudomonotone w.r.t. ψ.

(4) T is generalized hemicontinuous w.r.t. η $\Rightarrow$ T : generalized hemicontinuous w.r.t. some ψ.

Now we introduce a particular form of Theorem 1 in [21]; this is modified in order to achieve our main results. This theorem is a generalization of the well-known fixed point theorem of Fan-Browder(see Theorem 1 in [3]).

Theorem 2.1. Let X be a nonempty convex subset of a Hausdorff topological vector space E, K a nonempty compact subset of X. Let $A, B: X \to 2^X$ be two multifunctions. Suppose that

(1) for any $x \in X$, $Ax \subset Bx$;

(2) for any $x \in X$, Bx is convex;

(3) for any $x \in K$, $Ax \neq \emptyset$;

(4) for any $y \in X$, $A^{-1}y$ is open; and

(5) for each finite subset N of x, there exists a nonempty compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, $Ax \cap L_N \neq \emptyset$.

Then B has a fixed point $\bar{x}$, that is, $\bar{x} \in B\bar{x}$.

3. Implicit Vector Variational Inequalities

By Lemma 2.1 and Theorem 2.1, we obtain the following existence theorem of the implicit vector variational inequality (*IVVI*) under the generalized pseudomonotonicity condition.

Theorem 3.1. Let E be a Hausdorff topological vector space which the topological dual space E^* of E separates points on E, X a nonempty convex subset of E, F another topological vector space and $C: X \to 2^F$ a multifunction such that for each $x \in X$, Cx is a convex cone in F with $int\, Cx \neq \emptyset$ and $Cx \neq F$, $P := \bigcap_{x \in X} Cx$, $L(E, F)$ equipped with either the topology of pointwise convergence or the topology of bounded convergence, $\psi: L(E, F) \times X \times X \to F$ a function and $T: X \to 2^{L(E,F)}$ a multifunction. Let K be a nonempty weakly compact subset of X and $W: X \to 2^F$, $Wx = F \setminus (-int Cx)$, such that the graph $Gr(W)$ of W is weakly closed in $X \times F$. Assume that the following conditions are satisfied;

(1) T is generalized C-pseudomonotone w.r.t. ψ;

(2) T is generalized hemicontinuous w.r.t. ψ;

(3) for each $s \in L(E, F)$ and $x \in X$, $\psi(s, x, \cdot)$ is P-convex;

(4) for each $t \in L(E, F)$ and $x \in X$, $\psi(t, x, \cdot)$ is continuous where both

X and F are equipped with the weak topologies;

(5) for any $s \in L(E,F)$ and $x \in X$, $\psi(s,x,x) \in P$; and

(6) for each finite subset N of X, there exists a nonempty weakly compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there exists $y \in L_N$ such that there exists $t \in Ty$, $-\psi(t,y,x) \in -intCx$.

Then there exists $\bar{x} \in K$ such that $\bar{x}$ is a solution of the implicit vector variational inequality $(IVVI)$.

Proof. Define two multifunctions $A, B: X \to 2^X$ to be

$$Ax := \{y \in X \mid \exists t \in Ty, -\psi(t,y,x) \in -intCx\},$$

$$Bx := \{y \in X \mid \forall s \in Tx, \ \psi(s,x,y) \in -intCx\}.$$

The proof is organized in the following parts.

(i) Since T is generalized $C-$pseudomonotone w.r.t. ψ, for any $x \in X$, $Ax \subset Bx$.

(ii) For each $x \in X$, Bx is convex, Indeed, when y, $z \in Bx$ and $\alpha \in [0,1]$, we have for any $s \in Tx$,

$$\begin{aligned}\psi(s,x,\alpha y+(1-\alpha)z) &\in \alpha\psi(s,x,y)+(1-\alpha)\psi(s,x,z)-P\\ &\subset \alpha(-intCx)+(1-\alpha)(-intCx)-P\\ &\subset -intCx-Cx\\ &\subset -intCx.\end{aligned}$$

Hence $\alpha y+(1-\alpha)z \in Bx$, as desired.

(iii) For each $y \in X$, $A^{-1}y$ is weakly open. In fact, let $\{x_\lambda\}$ be a net in $(A^{-1}y)^c$ weakly convergent to $x \in X$. Then $y \notin Ax_\lambda$ and hence for any $t \in Ty$, $-\psi(t,y,x_\lambda) \notin -intCx_\lambda$. Thus for any $t \in Ty$, $-\psi(t,y,x_\lambda) \in Wx_\lambda$. Since $(x_\lambda, -\psi(t,y,x_\lambda)) \in Gr(W)$, by virtue of assumption (4) and the weak closedness of $Gr(W)$, $-\psi(t,y,x) \in Wx$ for any $t \in Ty$, that is, for any $t \in Ty$, $-\psi(t,y,x) \notin -intCx$, and hence $y \notin Ax$, namely, $x \in (A^{-1}y)^c$. Therefore $(A^{-1}y)^c$ is weakly closed, whence

$A^{-1}y$ is weakly open.

(iv) By hypothesis (6), for each finite subset N of X, there exists a nonempty weakly compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there exists $y \in L_N$ such that there exists $t \in Ty$, $-\psi(t, y, x) \in -intCx$. Thus for each $x \in L_N \setminus K$, there exists $y \in L_N$ such that $y \in Ax$ and hence $L_N \cap Ax \neq \emptyset$.

(v) B has no fixed point. If not, there exists $x \in X$ such that for any $s \in Tx$, $\psi(s, x, x) \in -intCx$ By assumption (5), for any $s \in Tx$, $\psi(s, x, x) \in (-intCx) \cap Cx = \emptyset$, which is a contradiction. Indeed, if there exists $v \in (-intCx) \cap Cx$, then $0 = -v + v \in -intCx + Cx = -intCx$ This implies $Cx = F$ because $0 \in intCx$ and $intCx$ is an absorbing set in F, which contradicts the assumption $Cx \neq F$. Therefore B has no fixed point.

(vi) From (i)-(v), we see, by Theorem 2.1, that there must be $\bar{x} \in K$ such that $A\bar{x} = \emptyset$, namely, for any $y \in X$, $y \notin Ax$, that is, for any $t \in Ty$,

$$-\psi(t, y, \bar{x}) \notin -intC\bar{x}. \tag{1}$$

Suppose to the contrary that $\bar{x}$ is not a solution of $(IVVI)$. Then there exists $\hat{y} \in X$ such that for any $s \in T\bar{x}$,

$$\psi(s, \bar{x}, \hat{y}) \in -intC\bar{x}. \tag{2}$$

Let $\overline{x_\alpha} := \bar{x} + \alpha(\hat{y} - \bar{x})$ for $\alpha \in [0,1]$. Since X is convex, $\overline{x_\alpha} \in X$.

Define a multifunction $H: [0,1] \to 2^F$ by for any $\alpha \in [0,1]$,

$H(\alpha) := \{\psi(s, \overline{x_\alpha}, \hat{y}) : s \in T(\overline{x_\alpha})\}$. Then, by (2), $H(0) \subset -intC\bar{x}$. Since T is generalized hemicontinuous w.r.t. ψ, there exists $\hat{\alpha} \in (0,1]$ such that for any $\alpha \in [0, \hat{\alpha})$, $H(\alpha) \subset -intC\bar{x}$. Hence there exists $\hat{\alpha} \in (0,1]$ such that for any $\alpha \in (0, \hat{\alpha})$ and $s \in T(\overline{x_\alpha})$,

$$\psi(s, \overline{x_\alpha}, \hat{y}) \in -intC\bar{x}. \tag{3}$$

Fix $\alpha \in (0, \hat{\alpha})$. By the P-convexity of $\psi(s, \overline{x_\alpha}, \cdot)$, we have for any $s \in T(x_\alpha)$,

$$\begin{aligned}\psi(s, \overline{x_\alpha}, \overline{x_\alpha}) &= \psi(s, \overline{x_\alpha}, \alpha\hat{y} + (1-\alpha)\bar{x}) \\ &\in \alpha\psi(s, \overline{x_\alpha}, \hat{y}) + (1-\alpha)\psi(s, \overline{x_\alpha}, \bar{x}) - P.\end{aligned}$$

From (3) and assumption (5), we have for any $s \in T(\overline{x_\alpha})$,

$$\begin{aligned}-(1-\alpha)\psi(s, \overline{x_\alpha}, \bar{x}) &\in \alpha\psi(s, \overline{x_\alpha}, \hat{y}) - \psi(s, \overline{x_\alpha}, \overline{x_\alpha}) - P \\ &\subset -intC\bar{x} - P - P \\ &\subset -intC\bar{x} - C\bar{x} - C\bar{x} \\ &\subset -intC\bar{x}.\end{aligned}$$

Thus for any $s \in T(\overline{x_\alpha})$, $-\psi(s, \overline{x_\alpha}, \bar{x}) \in -intC\bar{x}$, which contradicts (1). Hence $\bar{x}$ is a solution of $(IVVI)$.

From Lemma 2.1 and Theorem 3.1, we can easily obtain the following corollary.

Corollary 3.1. Let E, F, K, C, W and P be as in Theorem 3.1. Suppose that X is a nonempty bounded convex subset of E and $L(E, F)$ is equipped with the topology of bounded convergence. Let $\eta: X \times X \to E$ be a function. and $T: X \to 2^{L(E,F)}$ a multifunction. Assume that the following conditions are satisfied;

(1) T is generalized $C-$pseudomonotone w.r.t. η;

(2) T is generalized hemicontinuous w.r.t. η;

(3) for each $s \in L(E, F)$ and $x \in X$, $\langle s, \eta(\cdot, x)\rangle$ is $P-$convex;

(4) for each $x \in X$, $\eta(\cdot, x)$ is continuous where both X and F are equipped with the weak topologies;

(5) for any $s \in L(E, F)$ and $x \in X$, $\langle s, \eta(x, x)\rangle \in P$; and

(6) for each finite subset N of X, there exists a nonempty weakly compact

convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there exists $y \in L_N$ such that for any $t \in Ty$, $-\langle t, \eta(x,y) \rangle \in -intCx$.

Then there exists $\bar{x} \in K$ such that for each $y \in X$, there exists $s \in T\bar{x}$ such that $\langle s, \eta(y, \bar{x}) \rangle \notin -int C\bar{x}$.

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