APPROXIMATE CONTROLLABILITY AND REGULARITY FOR RETARDED SEMILINEAR CONTROL SYSTEM

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ABSTRACT.

This paper deals with the approximate controllability for semilinear system with time delay in Hilbert space. After the problem for existencs and uniqueness of solution of the given system with the more general Lipschitz continuity of nonlinear operator f from $\mathcal{R} \times V$ to H is established, it is shown that the equivalence between the reachable set of the semilinear system and that of its corresdonding linear system. Finally, we make a practical application of the condition to the system with only discrete delay.

1. INTRODUCTION

Let H be a Hilbert space and V be imbedded in H as a dense subspace. The dual space of V denoted by V^*. In this paper we deal with control problem for semilinear parabolic type equation in Hilbert space H as follows.

(1.1)

$$\begin{aligned} \frac{d}{dt}x(t) =& A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds \\ &+ f(t, x(t)) + Bu(t), \quad t \in (0, T]. \end{aligned}$$

Let A_0 generate an analytic semigroup in both H and V^*. Then the equation (1.1) may be considered as an equation in both H and V^*. Let the operators A_1 and A_2 be a bounded linear operators from V to V^* and $a(\cdot)$ be Hölder continous. The nonlinear operator f from

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$\mathcal{R} \times V$ to H is Lipschitz continuous. The first part of this paper is to give wellposedness and regularity in section 2. This approach is closed to that in [2,3] mentioned above. We will give the result by using the intermediate property and contraction mapping principle. Next, under more generalized the range condition of the controller than of in [6,8,9,15], we establish that the approximate controllability for semilinear system is equivalent to that of its corresponding linear system in section 3. It is known that the reachable set for linear sustem that is the set of all trajectories cerresponding to given control set is independent of the time. By proceeding to derived to the equivalent condition between semilinear and linear control system we can proof the approximate controllability of semilinear system as checking it at some time

There are many literature which deal with structural properties for the linear system(the case where $f = 0$) in [1,2,10,13] and S. Nakagiri [10] has dealt with structural properties and solution semigroups associated with (1.1). The controll proplem of general initial volue problem without delay term was discussed frequently as in [8,9,11,15]. With the aid of the solution semigroup and fundamental solution of (1.1) that was constructed in [13], the equation (1.1) can be also transformed onto an abstract equation

$$\frac{d}{dt}z(t) = Az(t) + F(z(t)) + Bu(t) \tag{1.2}$$

in the product space $Z = H \times L^2(-h, 0; V)$. Therefore we can also apply the result in [15] to this system, but we want to obtain more general conditions for retarded system (1.1) without time restriction in [15]. In [8, 9, 11, 14], the authors showed the approximately controllable under assumption that the nonlinear term $f(t, x(t))$ is uniformly bounded. The control problem of (1.2) that the semigroup generated by A_0 is compact operator was obtained by K. Naito [8,9] using topological degree theory. Now we note that the semigroup generated by A associated with the equation (1.1) is not compact operator and the generator A_0 is unbounded in general (see theorem 5.3 in [2]). But by virtue of the result in Aubin [1], by assumption the imbedding $D(A_0) \subset V$ is compact we will show that the solutuin mapping from admissable set to the set of all trajectories associared with control function is compact.

Thus, by topological degree theory we can obtain the equivalence conditions between the reachable trajectory set of the semilinear system and that of the associated with linear system. We also give an example that the the condition of nonlinear term can be checked in section 4.

2. Wellposedness and regularity

We consider the problem of control for the following retarded functional differential equation of parabolic type with nonlinear term

(2.1)
$$\frac{d}{dt}x(t) = A_0x(t) + A_2x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds + f(t,x(t)) + Bu(t),$$

(2.2) $$x(0) = g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0).$$

in Hilbert space in H. Let V be another Hilbert space such that $V \subset H \subset V^*$. Therefore, for the simplicity, we may regard that $||u||_* \le |u| \le ||u||$ for all $v \in V$ where the notations $|\cdot|$, $||\cdot||$ and $||\cdot||_*$ denote the norms of H, V and V^* respectively as usual. Let $a(u,v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding's inequality

(2.3) $$\text{Re } a(u,u) \ge c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \ge 0.$$

Let A_0 be the operator associated with a sesquilinear form

(2.4) $$(A_0u, v) = -a(u,v), \quad u, \ v \in V.$$

Then the operator A_0 is a bounded linear from V to V^*. Identifying the antidual of H with H we may consider $V \subset H \subset V^*$. The realization of A_0 in H which is the restriction of A_0 to

$$D(A_0) = \{u \in V : A_0u \in H\}$$

is also denoted by A_0. It is known that A_0 generates an analytic semigroup in both H and V^*. Assume that (2.3) holds for $c_1 = 0$ noting that $A_0 + c_1$ is an isomorphism from V to V^* if $c_1 \neq 0$.

We may assume that the imbeddong $D(A_0) \subset V$ is compact and $(D(A_0), H)_{1/2,2} = V$ satisfying

$$||u|| \leq C_1 ||u||^{1/2}_{D(A_0)} |u|^{1/2} \tag{2.5}$$

for some a constant $C_1 > 0$ where $(D(A_0), H)_{\theta,p}$ denotes the real interpolation space between $D(A_0)$ and H.

The operators A_1 and A_2 are bounded linear operators from V to V^* such that they map $D(A_0)$ into H. The function $a(\cdot)$ is assumed to be a real valued Hölder continous in $[-h, 0]$ and the controller operator B is a bounded linear operator from some Banach space U to H. Let f be a nonlinear mapping from $\mathcal{R} \times V$ into H. We assume that for any x_1, $x_2 \in V$ there exists a constant $L > 0$ such that

$$|f(t, x_1) - f(t, x_2)| \leq L||x_1 - x_2|| \tag{2.6}$$

$$f(t, 0) = 0. \tag{2.7}$$

LEMMA 2.1. *Let $T > 0$. Then*

$$H = \{x \in V^* : \int_0^T ||A_0 e^{tA_0} x||_*^2 dt < \infty\},$$

where $||\cdot||_*$ is the norm of the element of V^*. Therefore, we have that $H = (V, V^*)_{\frac{1}{2},2}$ satisfying

$$|u| \leq ||u|| ||u||_*$$

for every $u \in H$.

Proof. Put $u(t) = e^{tA_0}x$ for $x \in H$. From

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 = \text{Re } (\dot{u}(t), u(t)) = \text{Re } (A_0 u(t), u(t))$$
$$= -\text{Re } a(u(t), u(t)) \leq -c_0 ||u(t)||^2,$$

it follows

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + c_0||u(t)||^2 < 0.$$

By integrating over t yields

$$\frac{1}{2}|u(t)|^2 + c_0 \int_0^T ||u(s)||^2 ds \leq \frac{1}{2}|x|^2.$$

Hence, we obtain that

$$\int_0^T ||A_* e^{tA} x||_*^2 dt \leq \int_0^T ||u(s)||^2 ds < \infty.$$

Conversely, suppose that $x \in V^*$ and $\int_0^T ||A_0 e^{tA_0} x||_*^2 dt < \infty$. Put $u(t) = e^{tA_0}x$. Then since A_0 is an isomorphism from V to V^* there exists a constant $c > 0$ such that

$$\int_0^T ||u(t)||^2 dt \leq c \int_0^T ||A_0 u(t)||_*^2 dt = c \int_0^T ||A_0 e^{tA_0} x||_*^2 dt.$$

From the assumptions and $\dot{u}(t) = A_0 e^{tA_0} x$ it follows

$$u \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

Therefore, $x = u(0) \in H$.

By virtue of Lemma 2.1, replacing intermediate space F in the paper [2] with the space H, we can derive the results of G. Blasio, K. Kunisch and E. Sinestrari [2] regarding term by term to deduce the following result.

Proposition 2.1. *Let $g = (g^0, g^1) \in Z = H \times L^2(-h,0;V)$ and $f \in L^2(0,T;V^*)$. Then for each $T > 0$, a solution x of the equation (2.1) and (2.2) belongs to*

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

Moreover, for some constant C_T we have

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_T(|g^0| + ||g^1||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V^*)} + ||u||_{L^2(0,T;U)}),$$

where

$$||\cdot||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} = \max\{||\cdot||_{L^2(0,T;V)}, ||\cdot||_{W^{1,2}(0,T;V^*)}\}.$$

THEOREM 2.1. *Under the above assumptions (2.6) and (2.7) for the nonlinear mapping f, then there exists a unique solution x of (2.1) and (2.2) such that*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

for any $g=(g^0,g^1) \in Z = H \times L^2(-h,0;V)$. Moreover, there exists a constant C such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C(|g^0| + ||g^1||_{L^2(-h,0;V)} + ||u||_{L^2(0,T;U)}).$$

Proof. Let us fix $T \in (0,h)$ such that

$$C_1 C_T L(T/\sqrt{2})^{\frac{1}{2}} < 1. \tag{2.8}$$

For $i=1,\ 2$, we consider the following equation.

$$\begin{aligned} \frac{d}{dt}y_i(t) =& A_0 y_i(t) + A_1 y_i(t-h) + \int_{-h}^0 a(s)A_2 y_i(t+s)ds \\ &+ f(t,x_i(t)) + Bu(t), \qquad t \in (0,T] \\ y_i(0) =& g^0, \quad y_i(s) = g^1(s), \qquad s \in [-h,0). \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dt}(y_1(t)-y_2(t)) =& A_0(y_1(t)-y_2(t)) + A_1(y_1(t-h)-y_2(t-h)) \\ &+ \int_{-h}^0 a(s)A_2(y_1(t+s)-y_2(t+s))ds \\ &+ f(t,x_1(t)) - f(t,x_2(t)), \qquad t \in (0,T] \\ y_1(0)-y_2(0) =& 0, \quad y_1(s)-y_2(s) = 0, \qquad s \in [-h,0). \end{aligned}$$

From Theorem 3.3 of [2] and (2.6) it follows that

$$\begin{aligned} &||y_1 - y_2||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} \le C_T ||f(\cdot,x_1) - f(\cdot,x_2)||_{L^2(0,T;H)}, \\ &||f(\cdot,x_1) - f(\cdot,x_2)||_{L^2(0,T;H)} \le L||x_1 - x_2||_{L^2(0,T;V)}. \end{aligned}$$

Using the Hölder inequality we also obtain that

(2.9)

$$
\begin{aligned}
||y_1 - y_2||_{L^2(0,T;H)} &= \{\int_0^T |y_1(t) - y_2(t)|^2 dt\}^{\frac{1}{2}} \\
&= \{\int_0^T |\int_0^t (\dot{y}_1(\tau) - \dot{y}_2(\tau)) d\tau|^2 dt\}^{\frac{1}{2}} \\
&\le \{\int_0^T t \int_0^t |\dot{y}_1(\tau) - \dot{y}_2(\tau)|^2 d\tau dt\}^{\frac{1}{2}} \\
&\le \frac{\sqrt{T}}{2} ||y_1 - y_2||_{W^{1,2}(0,T;H)}.
\end{aligned}
$$

Therefore, in terms of (2.5) and (2.9) we have

$$
\begin{aligned}
||y_1 - y_2||_{L^2(0,T;V)} &\le C_1 ||y_1 - y_2||^{\frac{1}{2}}_{L^2(0,T;D(A_0))} ||y_1 - y_2||^{\frac{1}{2}}_{L^2(0,T;H)} \\
&\le C_1 ||y_1 - y_2||^{\frac{1}{2}}_{L^2(0,T;D(A_0))} (\frac{T}{\sqrt{2}})^{\frac{1}{2}} ||y_1 - y_2||^{\frac{1}{2}}_{W^{1,2}(0,T;H)} \\
&\le C_1 (\frac{T}{\sqrt{2}})^{\frac{1}{2}} ||y_1 - y_2||_{L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)} \\
&\le C_1 C_T (\frac{T}{\sqrt{2}})^{\frac{1}{2}} ||f(\cdot, x_1) - f(\cdot, x_2)||_{L^2(0,T:H)} \\
&\le C_1 C_T L (\frac{T}{\sqrt{2}})^{\frac{1}{2}} ||x_1 - x_2||_{L^2(0,T;V)}.
\end{aligned}
$$

So by virtue of the condition (2.8) the contraction principle gives that the equation of (2.1) and (2.2) has a unique solution in $[-h, T]$.

Let $x(\cdot)$ be a solution of (2.1) and (2.2) and $y(\cdot)$ be a solution of following equation.

$$
\begin{aligned}
\frac{d}{dt} y(t) =& A_0 y(t) + A_1 y(t-h) \int_{-h}^0 a(s) A_2 y(t+s) ds \\
&+ Bu(t), \qquad\qquad t \in (0, T] \\
y(0) =& g^0, \quad y(s) = g^1(s), \qquad s \in [-h, 0).
\end{aligned}
$$

Consider the following problem:

$$\begin{aligned}\frac{d}{dt}(x(t)-y(t)) =& A_0(x(t)-y(t))+A_1(x(t-h)-y(t-h)) \\ &+\int_{-h}^{0} a(s)A_2(x(t+s)-y(t+s))ds+f(t,x(t)), \\ x(0)-y(0) =& 0, \quad x(s)-y(s)=0 \qquad\qquad s\in[-h,0).\end{aligned}$$

In virtue of Theorem 3.3 of [1] we have

$$\begin{aligned}||x-y||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} &\le C_T||f(\cdot,x)||_{L^2(0,T;H)} \\ &\le C_T L||x||_{L^2(0,T:V)} \\ &\le C_T L\,(||x-y||_{L^2(0,T;V)}+||y||_{L^2(0,T;V)}).\end{aligned}$$

Combining (2.5), (2.9) and above inequality we have

$$\begin{aligned}||x-y||_{L^2(0,T;V)} &\le C_1||x-y||^{\frac{1}{2}}_{L^2(0,T;D(A_0))}||x-y||^{\frac{1}{2}}_{L^2(0,T;H)} \\ &\le C_1||x-y||^{\frac{1}{2}}_{L^2(0,T;D(A_0))}\{\frac{T}{\sqrt{2}}||x-y||_{W^{1,2}(0,T;H)}\}^{\frac{1}{2}} \\ &\le C_1(\frac{T}{\sqrt{2}})^{\frac{1}{2}}||x-y||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} \\ &\le C_1(\frac{T}{\sqrt{2}})^{\frac{1}{2}}C_T L(||x-y||_{L^2(0,T;V)}+||y||_{L^2(0,T;V)}).\end{aligned}$$

Therefore, we have

$$||x-y||_{L^2(0,T;V)} \le \frac{C_1C_TL(\frac{T}{\sqrt{2}})^{\frac{1}{2}}}{1-C_1C_TL(\frac{T}{\sqrt{2}})^{\frac{1}{2}}}||y||_{L^2(0,T;V)},$$

(2.10)

$$||x||_{L^2(0,T;V)} \le \frac{1}{1-C_1C_TL(\frac{T}{\sqrt{2}})^{\frac{1}{2}}}||y||_{L^2(0,T;V)}.$$

Combining Proposition 2.1 and (2.10) we obtain

$$
\begin{aligned}
&||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)}\\
\le & C_T(|g_0| + ||g^1||_{L^2(0,T;V)} + ||f(\cdot,x)||_{L^2(0,T;V^*)}\\
&\quad + ||u||_{L^2(0,T:U)})\\
\le & C_T(|g^0| + ||g^1||_{L^2(0,T;V)} + L||x||_{L^2(0,T;V)}\\
&\quad + ||u||_{L^2(0,T;U)})\\
\le & C_T(|g_0| + ||g^1||_{L^2(0,T;V)} + ||u||_{L^2(0,T:U)}\\
&\quad + \frac{L}{1 - C_1C_TL(\frac{T}{\sqrt{2}})^{\frac{1}{2}}}||y||_{L^2(0,T;V)})\\
\le & C_T(|g_0| + ||g^1||_{L^2(0,T;V)} + ||u||_{L^2(0,T:U)})\\
&\quad + \frac{LC_T}{1 - C_1C_TL(\frac{T}{\sqrt{2}})^{\frac{1}{2}}}(|g^0| + ||g^1||_{L^2(0,T;V)}\\
&\quad + ||u||_{L^2(0,T:U)})\\
\le & C(|g_0| + ||g^1||_{L^2(0,T;V)} + ||u||_{L^2(0,T:U)}).
\end{aligned}
$$

Since the condition (2.8) is independent of initial value, the solution of (2.1) and (2.2) can be extended to the interval $[-h, nT]$ for a natual number n, that is, we can prove the estimate mentioned above also in the interval $[T, 2T]$ with initial data $(x_T, x(T))$. So the proof is complete.

3. Approximate controllability for retarded system

Let $x(t; f, w)$ be a solution of the following equation associated with nonlinear term f and control function w at time T.

$$
(3.1)\qquad \begin{cases} x'(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds \\ \qquad\qquad f(t, x(t)) + (Bu)(t), \quad 0 \le t \le T, \\ x(0) = 0, \quad x(s) = 0 \quad -h \le s < 0, \end{cases}
$$

where B is bounded linear operator from $L^2(0,T;U)$ to $L^2(0,T;H)$.

We define reachable sets for the system (3.1) as follows:

$$R_T(0) = \{x(T;0,u) : u \in L^2(0,T;U)\},$$
$$R_T(f) = \{x(T;f,u) : u \in L^2(0,T;U)\}.$$

If $\overline{R_T(f)} = H$ where $\overline{R_T(f)}$ is the closure of $R_T(f)$ in H then the system (3.1) is called approximate controllable. Let $G(t)$ be an analytic semigroup generated by A_0. We now define the fundamental solution $W(t)$ of (3.1) by

$$\frac{d}{dt}W(t) = \begin{cases} A_0W(t) + A_1W(t-h) = \int_{-h}^{0} a(s)A_2W(t+s)ds, & 0 \le t \\ W(0) = I, \quad W(s) = 0 \quad s < [-h,0). \end{cases}$$

According to the above definition $W(t)$ is a unique solotion of

$$W(t) = G(t) + \int_0^t G(t-s)\{A_1W(s-h) + \int_{-h}^{0} a(\tau)A_2W(s+\tau)d\tau\}ds$$

for $t \geq 0$ (cf. S. Nakagiri [5]). The initial value problem (3.1) has a unique solution satisfying the integral equation. The solution of (3.1) is expressed by

$$x(t) = W(t)g^0 + \int_{-h}^{0} U_t(s)g^1(s)ds + \int_0^t W(t-\tau)f(\tau, x(\tau))d\tau,$$
$$U_t(s) = W(t-s-h)A_1 + \int_{-h}^{s} W(t-s+\sigma)a(\sigma)A_2d\sigma$$

Consider the following semilinear equation which is described by control system on H where the controller B is the identity operator:

$$(3.2) \qquad \begin{cases} x'(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds \\ \qquad f(t,x(t)) + u(t), \quad 0 \le t \le T, \\ x(0) = 0, \quad x(s) = 0 \quad -h \le s < 0, \end{cases}$$

The solution y_u is given by

$$y_u(t) = \int_0^t W(t-s)\{f(s, y_u(s)) + u(s)\}ds,$$

for $0 \le t \le T$.

We can define the nonlinear operator $\mathcal{F}$ on $L^2(0,T;H)$ by

$$(\mathcal{F}u)(t) = f(t, y_u(t)), \quad u \in L^2(0,T;H).$$

LEMMA 3.1. *Let* $f \in L^2(0,T;H)$ *and* $x(t) = \int_0^t W(t-s)f(s)ds$. *Then there exists a constant* C *such that*

$$||x||_{L^2(0,T;V)} \le C\sqrt{T}||f||_{L^2(0,T;H)}.$$

Proof. By the similary way of Theorem 2.3 of [2] it holds that

$$||x||_{L^2(0,T;D(A_0))} \le C_T||f||_{L^2(0,T;H)}. \tag{3.3}$$

By using Hölder inequality,

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t W(t-s)f(s)ds|^2 dt \\ &\le M^2 \int_0^T (\int_0^t |f(s)|ds)^2 dt \\ &\le M^2 \int_0^T t \int_0^t |f(s)|^2 dsdt \\ &\le M^2 \frac{T^2}{2} \int_0^t |f(s)|^2 ds. \end{aligned}$$

Therefore

$$||x||_{L^2(0,T;H)} \le MT||f||_{L^2(0,T;H)}. \tag{3.4}$$

Combining (3.3) and (3.4) we have that

$$||x||^2_{L^2(0,T;V)} \le C_T M\sqrt{T}||f||^2_{L^2(0,T;H)}.$$

LEMMA 3.2. *Let x_u be a solution of (3.2). Then for $T > 0$ there exists a constant C such that*

$$||\mathcal{F}u||_{L^2(0,T;H)} \leq LC\sqrt{T}/(1 - LC\sqrt{T})||u||_{L^2(0,T;H)}.$$

Proof. From Lemma 3.1 it follows that

$$\begin{aligned} ||\mathcal{F}u||_{L^2(0,T;H)} &\leq L||x_u||_{L^2(0,T;V)} \\ &\leq L||\int_0^t W(t-s)\{\mathcal{F}u)(s)\}ds||_{L^2_t(0,T;V)} \\ &\quad + L||\int_0^t W(t-s)\{u(s)\}ds||_{L^2_t(0,T;V)} \\ &\leq LC\sqrt{T}||\mathcal{F}u||_{L^2(0,T;H)} \\ &\quad + LC\sqrt{T}||u||_{L^2(0,T;H)} \end{aligned}$$

where we set $||f(t)||_{L^2_t(0,T;V)} = ||f||_{L^2(0,T;V)}$.

Let

$$N = \{p \in L^2(0,T;H); \int_0^T S(T-s)p(s)ds = 0\}$$

and its orthogonal space in $L^2(0,T;H)$ by $N^\perp$.

We denote the range of the operator B by H_B. We need the following assupmtion:

(A) Let H_B be closed and for each $p \in L^2(0,T;H)$ there exists $q \in H_B$ such that

$$\int_0^T W(T-s)p(s)ds = \int_0^T W(T-s)q(s)ds,$$

that is, $L^2(0,T;H) = H_B + N$.

As is seen in [4] it need not assume the range of the operator B is closed, but for the sake of simplicity we assume it in what follows. The meaningful of this assumption is considered in section 4.

Let P_J be the projection on $L^2(0,T;H)$ with the range $N^\perp$. For every $u \in N^\perp$, taking by Pu as the unique minimum norm element in $\{u + N\} \cap H_B$ we can define P as the mapping from $N^\perp$ to H_B. So, we know that $u - Pu \in N$ and $Pu \in H_B$. Let $\tilde{Y} = L^2(0,T;H)/N$ be the quotient space with norm $||\tilde{y}|| = \inf\{|y+f| : f \in N\}$. Let

$$\tilde{\mathcal{F}}\tilde{u} = P_J \cdot \mathcal{F} \cdot P\tilde{u}, \quad \tilde{u} \in N^{\perp}.$$

Then from Lemma 3.2 it follows

$$||\tilde{\mathcal{F}}\tilde{u}||_{L^2(0,T;H)} \le LC\sqrt{T}/(1 - LC\sqrt{T})||P||||u||_{L^2(0,T;H)}.$$

We will show that $\tilde{\mathcal{F}}$ is a compact operator.

THEOREM 3.1. *Under the above assumption (A), we have*

$$\overline{R_T(0)} = \overline{R_T(f)} = H.$$

Therefore, the system (3.1) is approximately controllable.

Proof. Under assumption (A) it is known that $\overline{R_T(0)} = H$ as is seen in [8]. Thus it suffices to prove that $R_T(0) \subset \overline{R_T(f)}$. The solution of (3.1) is

$$x(t;f,w) = \int_0^t W(t-s)\{f(s,x(s;f,w)) + (Bu)(s)\}ds,$$

and from Theorem 2.1 it follows

$$||x(t;f,w)||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C||u||_{L^2(0,T;U)}.$$

The estimate of solution of (3.2) defined by y_u is

$$||y_u||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C||u||_{L^2(0,T;H)}.$$

Since $y_u \in L^2(0,T;V)$ we have $f(\cdot,y_u) \in L^2(0,T;H)$, and hence $y_u \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$ by Theorem 3.2 in [2],

(3.6)
$$\begin{aligned} ||y_u||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} &\le C||f(\cdot,y_u) + u||_{L^2(0,T;H)}) \\ &\le C(||y_u||_{L^2(0,T;V)} + ||u||_{L^2(0,T;H)}) \le C||u||_{L^2(0,T;H)}. \end{aligned}$$

In virtue of Theorem 2 in [1], we know that imbeding $L^2(0,T;D(A))\cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$ is compact since imbedding $D(A) \subset V$ is compact. If u is bounded in $L^2(0,T;H)$ then from (3.5) it follows that y_u is also bounded in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$, and hence it is relatively compact in $L^2(0,T;V)$). Thus, the mapping $u \mapsto y_u$ is a compact operator from $L^2(0,T;U)$ to $L^2(0,T;H)$ and $\mathcal{F}$ is also a compact operator from $L^2(0,T;H)$ into itself.

Let $\phi \in R_T(0)$. Then by virtue of the asumption (A), there exists a control function $v \in L^2(0,T;U)$ such that

$$\phi = \int_0^T W(T-s)Bv(s)ds.$$

Take a time $T > 0$ suffidiently small such that

$$LC\sqrt{T}||P|| < \frac{1}{2} \tag{3.7}$$

where the constants are in Lemma 3.2. Put $\tilde{z} = P_J Bv$ and $C_1 = LC\sqrt{T}||P||$. Then we can take a constant such that

$$\frac{1-C_1}{1-2C_1}||\tilde{z}|| < R.$$

Let $U_R = \{\tilde{u} \in N^\perp : ||\tilde{u}||_U < R\}$ be the open ball in $N^\perp$. Then since $(1-C_1)/(1-2C_1) > 1$ it follows $\tilde{z} \in U_R$. Let us consider the equation

$$\tilde{z} = \lambda \tilde{\mathcal{F}}\tilde{u} + \tilde{u}, \quad 0 \le \lambda \le 1.$$

Then

$$\begin{aligned} ||\tilde{u}|| &\le ||\tilde{z}|| + ||\tilde{\mathcal{F}}\tilde{u}|| \\ &\le ||\tilde{z}|| + \frac{C_1}{1-C_1}||u||. \end{aligned}$$

Thus

$$||\tilde{u}|| \le \frac{1-C_1}{1-2C_1}||\tilde{z}|| < R,$$

that is, $\tilde{u} \notin \partial U_R$ for $0 \le \lambda \le 1$ where ∂U_R stsnds for the boundary of the open ball U_R. Thus by the homotopy property of degree theory there exists a control set $u \in U_R$ such that $\tilde{z} = \tilde{\mathcal{F}}\tilde{u} + \tilde{u}$. Put $u_B = P\tilde{u}$. Then

$$\tilde{z} = \mathcal{F}(u_B) + P_J u_B.$$

From that for every $p \in L^2(0,T;H)$

$$\int_0^t W(t-s)P_J p(s)ds = \int_0^t W(t-s)p(s)ds, \quad 0 \le t \le T$$

it follows that

$$\begin{aligned} \phi &= \int_0^T S(T-s)(\mathcal{F}(u_B)(s) + u_B(s)ds \\ &= \int_0^T S(T-s)(f(s, y_{u_B}(s) + u_B(s))ds. \end{aligned}$$

Since $u_B \in H_B$ there exists a control function $w \in L^2(0,T;U)$ such that $u_B = Bw$. Hence we obtain that

$$\phi = x(T; f, w).$$

Hence for sufficiently small T, we have proof that $\overline{R_T(0)} \subset \overline{R_T(g)}$. But since A_0 generates an analytic semigroup, $\overline{R_T(g)}$ is independent of the time T. Thus, we can apply this procedure on each interval $[(n-1)T, nT]$ where the time T satisfies (3.7) and n is natural number.

4. Examples

Let us consider the following equation

$$(4.1) \quad \begin{cases} x'(t) = A_0 x(t) + A_1 x(t-h) + f(t, x(t)) + (Bu)(t), & 0 \le t, \\ x(0) = 0, \quad x(s) = 0 & -h \le s < 0. \end{cases}$$

where the controller B is a bounded linear operator from $L^2(0,T;H)$ to itself. Then the fundermental solution of (4.1) is

$$(4.2) \quad W(t) = \sum_{j=0}^{n} \frac{(t-j)^j}{j!} A_1^j T(t-j) \quad t \in [n, n+1], \quad n = 1,\ 2,\ \ldots$$

which is not norm continuous at $t = 1,\ 2,\ \ldots.$

In what follows we assume the imbedding $V \subset H$ is compact and A_0 is a self adjoint operator; In virtue of the Riesz-Schauder theorem, if the imbedding $V \subset H$ is compact then the operator A_0 has discrete spectrum

$$\sigma(A_0) = \{\mu_n : n = 1,\ 2,\ \ldots\}$$

which has no point of accumulation except possibly $\mu = \infty$. Let μ_n be a pole of the resolvent of A_0 of order k_n and P_n the spectral projection associated with μ_n

$$P_n = \frac{1}{2\pi i}\int_{\Gamma_n}(\mu - A_0)^{-1}d\mu,$$

where Γ_n is a small circle centered at μ_n such that it surrounds no point of $\sigma(A_0)$ except μ_n. Then the generalized eigenspace corresponding to μ_n is given by

$$H_n = P_n H = \{P_n u : u \in H\},$$

and we have that from $P_n^2 = P_n$ and $H_n \subset V$ it follows that

$$P_n V = \{P_n u : u \in V\} = H_n.$$

Let us set

$$Q_n = \frac{1}{2\pi i}\int_{\Gamma_n}(\mu - \mu_n)(\mu - A_0)^{-1}d\mu.$$

Then we remark that $\dim H_n < \infty$ and

$$Q_n^i = \frac{1}{2\pi i}\int_{\Gamma_n}(\mu - \mu_n)^i(\mu - A_0)^{-1}d\mu.$$

It is also well known that $Q_n^{k_n} = 0$ (nilpotent) and $(A_0 - \mu_n)P_n = Q_n$. If $\mu_n \in \sigma(A_0)$ then we have the Laurent expansion for $R(\mu - A_0) \equiv (\mu - A_0)^{-1}$ at $\mu = \mu_n$ whose principal part (the part consisting of all the negative power of $(\mu - \mu_n)$) is a finite series:

$$R(\mu - A_0) = \frac{P_n}{\mu - \mu_n} + \sum_{i=1}^{k_n - 1}\frac{Q_n^i}{(\mu - \mu_n)^{i+1}} + R_0(\mu),$$

where $R_0(\mu)$ is a holomorphic part of $R(\mu - A_0)$ at $\mu = \mu_n$.

DEFINITION 4.1. *The system of the generized eigenspaces of* A_0 *is complete in* H *if* $\mathrm{Cl}\{\mathrm{span}\{H_n : n = 1,\ 2,\ \dots\}\} = H$ *where* Cl *denotes the closure in* H.

First of all, for the meaning of assumption (A) in section 3 we need to show the existence of controller satisfying

$$\mathrm{Cl}\{Bu : u \in L^2(0,T;U)\} \neq L^2(0,T;H).$$

In fact, Consider about the controler B_0 defined by

$$Bu(t) = \sum_{n=1}^{\infty} u_n(t),$$

where

$$u_n = \begin{cases} 0, & 0 \le t \le \frac{T}{n} \\ P_n u(t), & \frac{T}{n} < t \le T. \end{cases}$$

Hence we see that $u_1(t) \equiv 0$ and $u_n(t) \in \mathrm{Im}\, P_n$. By completion of generalized eigenspaces of A_0 we may write that $f(t) = \sum_{n=1}^{\infty} P_n f(t)$ for $f \in L^2(0,T;H)$. Let us choose $f \in L^2(0,T;H)$ satisfying

$$\int_0^T ||P_1 f(t)||^2 dt > 0.$$

Then since

$$\int_0^T ||f(t) - Bu(t)||^2 dt = \int_0^T \sum_{n=1}^{\infty} ||P_n(f(t) - Bu(t))||^2 dt$$
$$\ge \int_0^T ||P_1(f(t) - Bu(t))||^2 dt = \int_0^T ||P_1 f(t)||^2 dt > 0,$$

the statement mentioned above is reasonable.

PROPOSITION 4.1. *Let the system of the generalized eizenspaces of* A_0 *be complete. Then If for every* $p \in L^2(0,1;H)$ *there exists* $q \in H_B$ *such that*

$$\sum_{k=i}^{k_n-1} \int_0^1 e^{-\lambda_n s} \frac{(-s)^{k-i}}{(k-i)!} Q_n^k (p(s) - q(s)) ds = 0$$

for $n=1,\ 2,...,i=0,\ \ldots\ ,k_n-1$ *then the system (4.1) is approximately controllable.*

proof. Let $G(t)$ be the semigroup generated by A_0. Then we give an expression of the semigroup that

$$G(t)f=e^{\mu_n t}\sum_{i=1}^{k_n-1}\frac{t^i}{i!}Q_n^i f,\quad t\geq 0$$

for any $f\in P_nH$. From (4.2) it holds

$$\int_0^1 W(1-s)p(s)ds=\int_0^1 G(1-s)p(s)ds$$

$$=\sum_{n=1}^{\infty}\sum_{i=0}^{k_n-1}\frac{e^{\lambda_n t}}{i!}\int_0^1 e^{\lambda_n s}(t-s)^i Q_n^i p(s)ds$$

$$=\sum_{n=1}^{\infty}\sum_{i=0}^{k_n-1}\sum_{k=i}^{k_n-1}e^{\lambda_n t}\frac{t^i}{i!(k-i)!}\int_0^1 e^{-\lambda_n s}(-s)^{k-i}Q_n^k p(s)ds.$$

Since the function $t\mapsto e^{\lambda_n t}t^i$ is linear independent, the above equality is equivalent to the fact that

$$\sum_{k=i}^{k_n-1}\int_0^1 e^{-\lambda_n t}\frac{(-s)^{k-i}}{(k-i)!}Q_n^k p(s)ds$$

for $n=1,\ 2,...,i=0,\ \ldots\ ,k_n-1$. Thus, we can rewrite the assumption (A) in section 3 as our result.

Example 1. Let $1<\alpha<T<2$ and define a cutting controller at time α on $L^2(0,T;H)$ by

$$(Bu)(t)=\begin{cases}u(s), & 0<t\leq\alpha\\ 0, & \alpha<t<T.\end{cases}$$

From funfamental solution (4.2) we know that

$$\int_0^T W(T-s)p(s)ds=\int_0^1 G(T-s)p(s)ds$$

$$+\int_1^T (T-s-1)A_0G(T-s-1)p(s)ds$$

for $p \in L^2(0,T;H)$ Put

$$u(s) = \begin{cases} p(s), & 0 < s \le 1 \\ p(s+T-\alpha) & 1 < s \le \alpha \\ +\frac{1}{\alpha-1}G(s)\int_1^{T-\alpha} G(T-\alpha-r)p(r)dr, & \\ 0, & \alpha < s \le T. \end{cases}$$

Then it is immediately seen that the operator B satisfies the assumption (A).

Example 2. Let

$$H = L^2(0,\pi),\ V = H_0^1(0,\pi),\ V^* = H^{-1}(0,\pi),$$
$$a(u,v) = \int_0^\pi u(x)\overline{v(x)}dx$$

and

$$A_0 = d^2/dx^2 \quad \text{with} \quad D(A_0) = \{y \in L^2(0,\pi) : y(0) = y(\pi) = 0\}.$$

The eigenvalue and the eigenfunction of A_0 are $\lambda_n = n^2$ and $\phi_n(x) = \sqrt{2/\pi}\sin nx$, respectively. The solution of the following equation

$$\frac{d}{dt}x(t) = A_0x(t) + A_1x(t-1) + f(t)$$

with initial date 0 is

$$x(t) = \begin{cases} \int_0^t e^{(t-s)A_0}f(s)ds & 0 < t < 1 \\ \int_0^t e^{(t-s)A_0}f(s)ds & 1 < t < 2 \\ +A_0\int_0^t (t-s-1)e^{(t-s-1)A_0}f(s)ds. & \end{cases}$$

Hence,

$$x(2) = \int_0^2 e^{(2-s)A_0}f(s)ds + A_0\int_0^1 (1-s)e^{(1-s)A_0}f(s)ds.$$

Let $\xi \in D(A_0)$ and

$$f(s) = \begin{cases} 0 & 0 < s < 1 \\ \xi - (s-1)A_0\xi & 1 < t < 2. \end{cases}$$

Then it follows that $x(2) = \xi$. Thus

$$\begin{aligned} R_2(0) =& \{\int_0^2 W(t-s)f(s)ds : f \in L^2(0,2;H)\} \\ =& \{\int_0^2 e^{(2-s)A_0} f(s)ds + A_0 \int_0^1 (1-s)e^{(1-s)A_0} f(s)ds : \\ & f \in L^2(0,2;H)\} \end{aligned}$$

is a dense subspace. Put

$$a(s) = \sum_{n=1}^{\infty} a_n(s)\phi_n$$

Then since

$$\begin{aligned} & \int_0^2 W(t-s)a(s)ds \\ & = \int_0^1 \sum_{n=1}^{\infty} a_n(s)e^{-(2-s)\lambda_n}\phi_n ds + \int_1^2 \sum_{n=1}^{\infty} a_n(s)e^{-(2-s)\lambda_n}\phi_n ds \\ & - \int_0^1 \sum_{n=1}^{\infty} a_n(s)(1-s)\lambda_n e^{-(1-s)\lambda_n}\phi_n ds \\ & = \int_0^1 \sum_{n=1}^{\infty} a_n(s)\{e^{-(2-s)\lambda_n} - \lambda_n(1-s)e^{-(1-s)\lambda_n}\}ds\phi_n \\ & + \int_1^2 \sum_{n=1}^{\infty} a_n(s)e^{-(2-s)\lambda_n} ds\phi_n \\ & = \sum_{n=1}^{\infty}[\int_0^1 a_n(s)\{e^{-(2-s)\lambda_n} - \lambda_n(1-s)e^{-(1-s)\lambda_n}\}ds \\ & + \int_1^2 a_n(s)e^{-(2-s)\lambda_n} ds]\phi_n \end{aligned}$$

it holds that

$$\int_0^2 W(t-s)a(s)ds = 0$$

if and only if

$$\int_0^1 a_n(s)\{e^{-(2-s)\lambda_n}-\lambda_n(1-s)e^{-(1-s)\lambda_n}\}ds+\int_1^2 a_n(s)e^{-(2-s)\lambda_n}ds=0.$$

Therefore, the assumption (A) is equivalent that for every $p(\cdot) = \sum_{n=1}^{\infty} p_n(\cdot)\phi_n \in L^2(0,2;H)$ such that $q(\cdot) = \sum_{n=1}^{\infty} q_n(\cdot)\phi_n \in H_B$ such that $p_n - q_n$ is orthogonal to

$$\psi_n(s) = \begin{cases} e^{\lambda_n s} - \lambda_n(1-s)e^{\lambda_n(s+1)} & 0 < s < 1 \\ e^{\lambda_n}s & 1 < t < 2 \end{cases}$$

for every n.

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