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Applications of Himmelberg's Fixed Point Theorems to Fuzzy Mappings

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1. Introduction

In a series of recent papers, Chang[1,2,3,4] developed many theorems on fixed point theory and on variational inequalities into fuzzy setting, from a theoretical point of view, in a variety of situations.

In this paper, along the same lines, we present three applications of the well-known Himmelberg fixed point theorem [5] to the existence of cyclical coincidences, and of the equilibrium points of generalized games and of solutions of generalized quasi-variational inequalities for fuzzy mappings.

2. Preliminaries

For the terminologies and notations, we mainly refer to [4] and [9]. In this paper multifunctions and fuzzy mappings are always denoted by capital letters and single valued functions are denoted by small letters. For topological spaces X and Y, a multifunction $F : X \to Y$ is said to be *upper semicontinuous*(u.s.c.) provided for each open subset V of Y, we have $\{x \in X | Fx \subset V\}$ is open in X; and *lower semicontinuous*(l.s.c.) provided for each open subset V of Y, we have $\{x \in X | Fx \cap V \neq \phi\}$ is open in X. Also, F is *continuous* if F is u.s.c. and l.s.c.. F is declared *compact* if the range $F(X)$ is contained in a compact subset of Y.

On the other hand, a mapping T from X into $F(Y)$ the collection of all fuzzy sets over Y is called *a fuzzy mapping* over X. If T is a fuzzy mapping over X then $T(x)$ (denoted by T_x in the sequel) is a fuzzy set over Y, and $T_x(y)$ is the degree of membership of the point y in T_x. When Y has linear structure, the fuzzy mapping T over X is said to be *convex* provided for any $x \in X$ the fuzzy set T_x is convex, i.e. for any $t \in [0,1]$ and any $y, z \in Y$ it is true that $T_x(ty + (1-t)z) \geq min\{T_x(y), T_x(z)\}$. The fuzzy mapping T is *closed* if and only if the membership function $T_x(y)$ is u.s.c. over $X \times Y$ (as a real function). Let $A \in F(Y), \alpha \in (0,1]$. Then the set

$$(A)_\alpha = \{y \in Y | A(y) \geq \alpha\}$$

is called *a α - cut set of A.*

From now on, otherwise specifically mentioned, all topological spaces are assumed to be Hausdorff, E denotes a real Hausdorff locally convex space, and E^* its topological dual space equipped with the strong topology. For a nonempty subset Y of E, $cc(Y)$ denotes the set of all nonempty closed convex subsets of E contained in Y. Let X be nonempty convex subset of E. A real valued function $f : X \to R$ is said to be *quasi-concave*, if for every real number t, the set $\{x \in X | f(x) \geq t\}$ is convex.

The following result is our starting point.

Proposition 1. (Himmelberg [5,Theorem 2]) Let X be a nonempty convex subset of a Hausdorff locally convex space E. Let $F : X \to cc(X)$ be a compact u.s.c. multifunction. Then F has a fixed point.

Using this proposition, we can prove Proposition 2 and 3 in the following Section 3.

3. Main Result

We begin with a non-compact version of Simon's result [12, Theorem 2.5].

Theorem 1. Let n be a positive integer and, for each $i \in Z_n$, let X_i be a nonempty convex subset of a locally convex space E_i and $T_i : X_i \to cc(X_{i+1})$ a compact u.s.c. multifunction. Then there exists $(x_0, x_1, \cdots, x_{n-1}) \in X_0 \times \cdots \times X_{n-1}$ such that for all $i \in Z_n, x_{i+1} \in T_i x_i$. Here $Z_n = \{0, 1, \cdots, n-1\}$ stands for the additive group modulo n.

Proof. In case when $n = 1$ Theorem 1 is essentially Proposition 1. We assume that $n \geq 2$. Let $X = X_0 \times \cdots \times X_{n-1}, E = E_0 \times \cdots \times E_{n-1}$ and define $T : X \to 2^X$ by

$$T(x_0, x_1, \cdots x_{n-1}) = T_{n-1}x_{n-1} \times T_0 x_0 \times \cdots \times T_{n-2}x_{n-2}$$

for $(x_0, x_1, \cdots, x_{n-1}) \in X$. Then T is a compact u.s.c. multifunction with nonempty compact convex values. Indeed, each T_i is u.s.c. and compact convex valued. Hence the product map T is u.s.c. and compact convex valued by virtue of Lassonde [10,Proposition 1 (5)]. It remains to show that T is compact. But this is also immediate because T_i is compact. By Proposition 1, T has a fixed point $(x_0, x_1, \cdots, x_{n-1}) \in X$, i.e. $(x_0, x_1, \cdots, x_{n-1}) \in T(x_0, x_1, \cdots, x_{n-1})$. This gives the required result.

This type of $(x_0, \cdots, x_{n-1})$ is called a *a cyclical coincidence point* in Simons [12]. Now we are ready to give our first main result about the existence of cyclical coincidences for fuzzy mappings.

Theorem 2. Let X_i and E_i be as in Theorem 1. Let K_i be a nonempty compact subset of X_i and $T^i : X_i \to F(K_{i+1})$ a closed convex fuzzy mapping for $i \in Z_n$. Suppose that for each i, there exists a l.s.c. function $\alpha_i : X_i \to (0.1]$ such that for any $x \in X_i$ the cut set $(T^i_x)_{\alpha_i(x)} = \{y \in K_{i+1} | (T^i_x)(y) \geq$

$\alpha_i(x)\}$ is a nonempty subset of K_{i+1}. Then there exists $(x_0, x_1, \cdots, x_{n-1}) \in X_0 \times \cdots \times X_{n-1}$ such that $T^i_{x_i}(x_{i+1}) \geq \alpha_i(x_i)$ for all $i \in Z_n$.

Proof. Define $\widehat{T}_i : X_i \to 2^{K_{i+1}}$ by $\widehat{T}_i(x) = (T^i_x)_{\alpha_i(x)}$ for all $x \in X_i$.

Claim 1. $\widehat{T}_i$ is closed convex-valued, hence compact convex-valued.

In fact, for any $y, z \in (T^i_x)_{\alpha_i(x)}$ and any $t \in [0.1]$ we have

$$T^i_x(ty + (1-t)z) \geq min\{(T^i_x)(y), (T^i_x)(z)\} \geq \alpha_i(x).$$

This implies that $ty + (1-t)z \in (T^i_x)_{\alpha_i(x)}$, that is, $(T^i_x)_{\alpha_i(x)}$ is convex. Next let $\{y_j\}_{j\in J}$ be a net of $(T^i_x)_{\alpha_i(x)}$ convergent to $y_0 \in K_{i+1}$. Obviously, $(x, y_j) \to (x, y_0)$.

As T^i is closed, we have

$$(T^i_x)(y_0) \geq \limsup_j (T^i_x)(y_j) \geq \alpha_i(x).$$

This show that $y_0 \in (T^i_x)_{\alpha_i(x)}$, i.e. $\widehat{T}_i(x)$ is closed.

Claim 2. $\widehat{T}_i$ is u.s.c..

It suffices to show that the set

$$\text{graph}(\widehat{T}_i) = \cup_{x\in X_i}\{(x,y)|u \in \widehat{T}_i(x)\}$$

is closed in $X_i \times K_{i+1}$ by means of Lassonde [10,Proposition 1 (2)]. Let $(x_j, y_j)_{j\in J}$ be a net of graph$(\widehat{T}_i)$ and $(x_j, y_j) \to (x_0, y_0)$ in $X_i \times K_{i+1}$. Since T^i is a closed fuzzy mapping, we have

$$(T^i_{x_0})(y_0) \geq \limsup_j (T^i_{x_j})(y_j) \geq \limsup_j \alpha_i(x_j) \geq \liminf_j \alpha_i(x_j) \geq \alpha_i(x_0).$$

Hence $y_0 \in \widehat{T}_i(x_0)$, i.e. $(x_0, y_0) \in \text{graph}(\widehat{T}_i)$, as desired.

It is obvious that $\widehat{T}_i$ is compact because $\widehat{T}_i(X_i) \subset K_{i+1}$. Applying Theorem 1, we conclude that there exits $(x_0, x_1, \cdots, x_{n-1}) \in X_0 \times \cdots \times X_{n-1}$ such that $x_{i+1} \in \widehat{T}_i(x_i)$, i.e. $T^i_{x_i}(x_{i+1} \geq \alpha_i(x_i)$ for all $i \in Z_n$. This completes the proof.

In case when $n = 1$, Theorem 2 reduces to the following.

Corollary 1. Let X be a nonempty convex subset of a locally convex space E. Let K be a nonempty compact subset of X and $T : X \to F(K)$ a closed convex fuzzy mapping. Suppose that there exists a l.s.c. function $\alpha : X \to (0, 1]$ such that for any $x_0 \in X$ such that $T_{x_0}(x_0) \geq \alpha(x_0)$.

Remark. If X is compact and $K = X$, Corollary 1 is due to Chang [2, Theorem 5 (i)]. So Corollary 1 is a generalization of Chang's result to non-compact case. Also Corollary 1 is a fuzzy version of Himmelberg's fixed point theorem [Proposition 1]. When $n = 2$, Theorem 2 is an interesting result.

Our second result is concerned with the existence of equilibrium points in generalized games. A generalized game is a game in which the choices of players cannot be made independently: each player must select a strategy in a subset determined by the strategies chosen by the other players. For the sake of completeness, we introduce the following which is induced by Proposition 1.

Proposition 2. (Kum [8, Theorem 6.2.2], Kim [7, Theorem 2]) Let $\{X_i\}_{i \in I}$ be an indexed family of nonempty convex subsets each in a locally convex space E_i and $\{K_i\}_{i \in I}$ a correspondingly index family of nonempty compact subsets of X_i's. For each $i \in I$, let $f_i : X = \prod_{i \in I} X_i \to R$ be a continuous function and $T_i : X^i = \prod_{j \in I, j \neq i} X_j \to cc(K_i)$ be a continuous multifunction such that $f_i(x^i, \cdot)$ is quasiconcave for all $x^i \in X^i$. Then there exists a point $u \in X$ such that $u_i \in T_i(u^i)$, $f_i(u) = \max_{y \in T_i(u^i)} f_i(u^i, y)$ for all $i \in I$.

Theorem 3. Let X_i, K_i and f_i be as in Proposition 2. Let $T^i : X^i \to F(K_i)$ be a closed convex fuzzy mapping for all $i \in I$. Suppose that for each i, there exists a l.s.c. function $\alpha_i : X^i \to (0,1]$ such that for any $x^i \in X^i$, the cut set $(T^i_{x^i})_{\alpha_i(x^i)} = \{y \in K_i | T^i_{x^i}(y) \geq \alpha_i(x^i)\}$ is nonempty and the multifunction $x^i \mapsto (T^i_{x^i})_{\alpha(x^i)}$ is l.s.c.. Then there exists a point $u \in X$ such that

$$T^i_{u^i}(u_i) \geq \alpha_i(u^i) \text{ and } f_i(u) = \max_{y : T^i_{u^i}(y) \geq \alpha_i(u^i)} f_i(u^i, y).$$

Proof. We can prove in the same way as in Theorem 2 that for each i, the multifunction $\widehat{T}_i : X^i \to cc(K_i)$ defined by $\widehat{T}_i(x^i) = (T_{x^i} i)_{\alpha_i(x^i)}$ is compact u.s.c.. Moreover $\widehat{T}_i$ is also l.s.c. by given conditions, whence $\widehat{T}_i$ continuous. Therefore all conditions in Proposition 2 are satisfied, so there exists an $u \in X$ such that $u_i \in \widehat{T}_i(u^i)$, $f_i(u) = \max_{y \in \widehat{T}_i(u^i)} f_i(u^i, y)$. This means that

$$T^i_{u^i}(u_i) \geq \alpha_i(u^i) \text{ and } f_i(u) = \max_{y : T^i_{u_i}(y) \geq \alpha_i(u^i)} f_i(u^i . y).$$

This completes the proof.

Now we are going to present our last result on generalized quasi-variational inequalities for fuzzy mappings. In the first place, we introduce the following.

Proposition 3. (Kum [9., Theorems 3 and 4]) Let X be a nonempty bounded convex subset of a locally convex space E. Let $S : X \to cc(X)$ be compact continuous and $T : X \to cc(E^*)$ compact u.s.c. Then there exist an $x_0 \in Sx_0$ and a $y_0 \in Tx_0$ such that

$$< y_0, x_0 - x > \leq 0 \qquad \text{for all} \qquad x \in Sx_0.$$

In particular, if X is a normed linear space, the condition that X is bounded is superfluous.

Remark. The above Proposition is a generalization of Shih and Tan[11, Theorem 4] and Kim [6] under non-compact setting. Of course, Proposition 3 is obtained by using Proposition 1.

Theorem 4. Let X, E and E^* be as in Proposition 3. Let K and L be nonempty compact subsets of X and E^* respectively. Let $S :\to F(K)$ and $T : X \to F(L)$ be closed convex fuzzy mappings. Suppose that there exist two l.s.c. functions $\alpha : X \to (0,1]$ and $\beta : X \to (0,1]$ such that for any $x \in X$ the cut sets $(S_x)_{\alpha(x)}$ and $(T_x)_{\beta(x)}$ are nonempty. Assume further that the multifunction $x \mapsto (S_x)_{\alpha(x)}$ is l.s.c. on X. Then there exist an $x_0 \in X$ and $y_0 \in L$ such that $S_{x_0}(x_0) \geq \alpha(x_0)$, $T_{x_0}(y_0) \geq \beta(x_0)$ and

$$< y_0, x_0 - x > \leq 0 \quad \text{for all} \quad x \quad \text{with} \quad S_{x_0}(x) \geq \alpha(x_0).$$

Proof. As we have seen in Theorems 2 and 3, we know that two multifunctions $\widehat{S} : X \to cc(K)$ and $\widehat{T} : X \to cc(L)$ defined by

$$\widehat{S}(x) = (S_x)_{\alpha(x)}, \qquad \widehat{T}(x) = (T_x)_{\beta(x)}$$

are compact u.s.c.. By hypothesis, $\widehat{S}$ is continuous. Thus all conditions of Proposition 3 are fulfilled. Hence there exist an $x_0 \in \widehat{S}x_0$ and a $y_0 \in \widehat{T}x_0$ such that

$$< y_0, x_0 - x > \leq 0 \qquad \text{for all} \qquad x \in \widehat{S}x_0.$$

This (x_0, y_0) is a desired one.

Remark. We would like to point out the difference between Theorem 4 and Chang and Zhu [4, Theorem 1]: (i) in [4], T is assumed to be monotone together with some kind of lower semicontinuity whereas T is closed convex without monotonicity in Theorem 4; (ii) the interacting set $\sum$ in Theorem 1 of [4] is no longer required to be open. But we imposed a stronger continuity condition on S, i.e. $\widehat{S}$ is l.s.c.;(iii) the domain X need not be compact in Theorem 4.

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