A note on cs-semistratifiable spaces

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cs-semistratifiable한 空間에 關하여

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本 論文에서는 cs-semistratifiable 空間을 導入해서,

cs-semistratifiable 空間은 모든 点이 G_δ인 空間에 包含됨을 論하고, cs-semistratifiable 空間은 收斂하는 數列이 一様하게 G_δ인 空間이기 위한 必要充分條件임을 보이고, 司空[7]에서는 compact cs-semistratifiable 空間이 metrizable임을 보였는데 여기서는 ordinal space$[0, \Omega]$가 compact이지만 cs-semistratifiable 空間이 아니므로 metrizable이 안됨을 證明하였다.

Abstract

In this paper, a new clase of spaces called cs-semistratifiable spaces is discussed. It is shown that this clase of spaces is contained in spaces in which every point is G_δ, and that T_2-space X is cs-semistratifiable if and only if X is a space in which every convergent sequences is uniformly G_δ. Sakong [7] had shown that compact cs-semistratifiable space is metrizable; however I would like to show that the ordinal space$[0, \Omega]$ is a compact space,but not a cs-semistratifiable space, and therefore it is non-metrizable.

Definition 1. A cs-semistratification for a topological space X is defined as a mapping g from $N \times X$ to the topology of X which satisfies the following conditions;

cs-1. $x \in g(n, x)$.

cs-2. $g(n+1, x) \subset g(n, x)$, and

cs-3. If a sequence $\{x_n\}$ converge to a unique point x, then $\bigcap_{i=1}^{\infty} g(i, \langle x; x_n \rangle) = \langle x; x_n \rangle$. Here, we used the notation$\langle x; x_n \rangle = \{x\} \cup \langle x_n \rangle$, where$\langle x_n \rangle$ denotes the range of the sequence$\{x_n\}$. Also we introduce $g(n, S) = \bigcup \{g(n, x); x \in S\}$ for any subset S of X.

From now on, spaces will mean only T_2-spaces. By definitions and Martin's result, it is clear that semistratifiable→c-semistratifiable→cs-semistratifiable.

We show that we have another theorem of cs-semistratifiable spaces.

Theorem 1. If X is cs-semistratifiable, then every point is G_δ.

Proof. Assume that X is cs-semistratifiable space, then there exists $g(n,x)$ such that x belongs to $g(n,x)$. Let x_n be equal to x, for infinitely many n and x_n be equal to unique point x.

If $x_1, x_2, x_3 \cdots\cdots$ converge to y then $x=y$, thus T_2 is unnecessary. Meanwhile X is T_1-space, so $\{y\}$ is closed. Hence $\{x\}=\cap g(n,\langle x, x_n\rangle)$.

Thus this complates the proof.

Theorem 2. X is cs-semistratifiable space if and only if X is a space in which every convergent sequences is uniformly G_δ.

Proof; Let X be cs-semistratifiable space, then $\{x_n\}$ converges to x.

Define $\cup_k(x, x_n)=g(k;\langle x;x_n\rangle)$. Then $\langle x, x_n\rangle$ is equal to $\bigcap_k \cup_k(x_1, x_n)$.

If $x_n\rightarrow x$, $y_n\rightarrow y$ s.t. $\langle y, y_n\rangle\subset\langle x, x_n\rangle$ then $\cup_k(y, y_n)\subset\cup_k(x, x_n)$ for each k. Conversely for each n convergent sequence x_n converges to x, there corresponds a sequence of open sets $\{\cup k(x, x_n)\mid k=1, 2\cdots\}$ such that $\langle x, x_n\rangle=\bigcap_k \cup_k(x, x_n)$ and that if $x_n\rightarrow x$, $y_n\rightarrow y$ with $\langle y;y_n\rangle\subset\langle x;x_n\rangle$, then. $\cup_k(y, y_n)$ is included in $\cup_k(x, x_n)$ for each k.

Define $g(k, x)$ to be $\cup_k(x)$. Where $\{\cup_k(x)\mid k=1, 2, \cdots\}$ is the sequence of open sets assigned to the convergent sequence $\{x, x, x, \cdots\}$ then g is a cs-semistratification, so the proof is complete.

A space X is said to be developable if it has a sequence of open covers$(r_1, r_2, r_3, \cdots)$ of X such that if $x_n\in \mathrm{st}(x, r_n)$ for each n, the sequence $\{x_n\}$ converges to x. A regular developable space is called a Moore space[2]. A space X is $\omega\Delta$-space if it has a sequence $(r_1, r_2, \cdots)$ of open covers of X such that if $x_n\in \mathrm{st}(x, r_n)$ for each n, the sequence $\{x_n\}$ has a chuster point. Similarly, a space is a ωM-spase [4] if it has a sequence$(r_1, r_2, \cdots)$ of open covers of X such that if $x\in \mathrm{st}^2(x, r_n)$ for each n, the sequence$\{x_n\}$ has a cluster point. Clearly every developable space and every ωM-space is a $\omega\Delta$-space.

A topological space X is a β-space provided that there is a mapping a from $N\times X$ to the topology of X such that $x\in g(n, x)$ for all n and all x and if $x\in g(n, x_n)$ for some $x\in X$ and a sequence$\{x_n\}$ in X, then$\{x_n\}$ has a cluster point. Martin [6] proves that a regular space is semistrafiable if and only if it is a c-semistratifiable β-space. It remains true when c- semistratifiability is replaced by cs-semistratifiability.

Lemma 1. A compact cs- semistraifiable space is metrizable[7]

Definition 2. X is Frechet if and only if every x belongs to $\bar{A}$ then there exists sequence $\{x_n\}$ in A s.t. $x_n\rightarrow x$.

Theorem 3. If X is 1^{st} countable space, then X is Frechét.

Proof; Let $x\in \bar{A}$ and $\{g(\mathrm{n}, x)\mid n\in Z^+\}$ be a decreasing countable local base at x.
$\forall_n$, $g(n, x)\cap \mathrm{A}\neq 0$, $x_n\in g(n, x)$, $x_n\rightarrow x \& x_n\in \mathrm{A}$.

Lemma 2. A regular space is semistratifiable if and only if it is a cs-semistratifiable β-spaces[7].

Theorem 4. [$0, \Omega$] is not Frechét(and hence not 1°-countable.)

Proof. Ω belongs to[$0, \Omega$], even if we take $\sup\{\alpha_n | \alpha_n < \Omega\} < \Omega$, α_n are a countable set but Ω is a uncountable set. Hence α_n dose not converge to Ω.

Theorem 5. If X is compact space, then X is developable (modk).

Proof. X is developable(modk), then there exists $(X, K; r)$, $\forall x \in K \subset U$. $\exists x \in K \subset U \rightarrow \exists n$, $st(x, r_n) \subset U$

Let $K = \{X\}, r_1, r_2, r_2, \cdots = \{X\}$. $\forall x \in X \subset X \rightarrow \exists n$, $st(x, \{x_n\}) \subset X$.

With the aid of theorem 4 and 5, I derive a basic property for ordinal space [$0, \Omega$].

Corollary. Compact space [$0, \Omega$] is developable (modk).

⟨References⟩

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