AN APPROACH TO ERDÖS' PROBLEM ON SUBSET-SUM-DISTINCT SEQUENCES

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Abstract. We use Tomić's inequality to show some interesting properties of subset-sum-distinct sequences. We reproduce L. Moser's result by means of an analytic method. Also we suggest a possible approach towards Erdös' conjecture on the lower bound of the n-th element of a subset-sum-distinct sequence.

1. Introduction

In this paper, by a *sequence* we mean a *strictly increasing sequence of positive integers.* We start with a definition.

Definition 1.1.

(i) Let A be a set of real numbers. We say that A has the subset-sum-distinct property (briefly SSD-property) if for any two finite subsets X, Y of A,

$$\sum_{x\in X} x = \sum_{y\in Y} y \quad \Longrightarrow \quad X = Y$$

Also, we say that A is SSD or A is an SSD-set if it has the SSD-property.

(ii) A sequence $\{a_n\}_{n=1}^{\infty}$ is called a subset-sum-distinct sequence (or briefly, an SSD-sequence) if it has the SSD-property.

One of the most interesting and natural SSD-sequences is $\mathbf{t} = \{1, 2, 2^2, 2^3, \cdots\}$. Now, for a given SSD-sequence $\{a_n\}_{n=1}^{\infty}$, how one can compare the size of this sequence with $\mathbf{t}$? The following basic lemma, which is a starting point of the next section, gives some insight.

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Lemma 1.2. *Let* $\{a_n\}_{n=1}^{\infty}$ *be an SSD-sequence. Then*

$$a_1 + a_2 + \cdots + a_n \;\geq\; 2^n - 1$$

for every $n \geq 1$.

Proof. Let

$$A = \{a_1, a_2, \cdots, a_n\} \quad \text{and} \quad J = \left\{ \sum_{b \in B} b \,:\, \phi \neq B \subset A \right\}.$$

Note that all the elements of J are positive integers. Since A has the SSD-property,

$$B, B' \subset A \quad \text{and} \quad B \neq B' \quad \Longrightarrow \quad \sum_{b \in B} b \neq \sum_{b' \in B'} b'$$

Hence $|J| = 2^n - 1$. Because $a_1 + a_2 + \cdots + a_n$ is the greatest element in J, we have

$$a_1 + a_2 + \cdots + a_n \;\geq\; 2^n - 1.$$

□

As $\{1, 2, 2^2, 2^3, \cdots\}$ suggests, SSD-sequences are quite sparce. It seems very natural to ask how dense they can be. We will consider a question of this flavor in this paper.

As a way of obtaining finite "dense" SSD-sets, one can use the Conway-Guy sequence (see [9], [11]). Here we explain the construction of the Conway-Guy sequence. First, define an auxiliary sequence u_n by

$$u_0 = 0\,, \quad u_1 = 1 \qquad \text{and} \qquad u_{n+1} = 2\,u_n - u_{n-r}\,, \quad n \geq 1\,,$$

where $r = \langle \sqrt{2n} \rangle$, the nearest integer to $\sqrt{2n}$. Now, for a given positive integer n, we define

$$a_i = u_n - u_{n-i}\,, \quad 1 \leq i \leq n\,.$$

The well-known Conway-Guy conjecture is that $\{a_i : 1 \leq i \leq n\}$ is SSD for any positive integer n. This is still open. It is known that it does yield SSD-sets for $n < 80$ (see [11, p.307, Theorem4.6]).

Throughout the paper, we assume that $\{a_n\}_{n=1}^{\infty}$ is an SSD-sequence.

(i) In the next section, we consider the Dirichlet series of an SSD-sequence, namely, $\sum_{i=1}^{m} a_i{}^s$ for real s.

(ii) In the third section, we consider a lower bound on a_n. Here a result of L. Moser asserts that

$$\sum_{i=1}^{m} a_i{}^2 \le \sum_{i=1}^{m} 2^{2(i-1)}.$$

We introduce a new approach to this inequality based on Laplace's method for estimating integrals, and suggest how this method might be further utilized.

2. Tomić's Inequality

First of all, we introduce Tomić's inequality. It will turn out that his result is extremely useful for the estimation of $\sum_{i=1}^{m} a_i{}^s$, especially when s is not an integer.

Theorem 2.1. (M. Tomić, 1949) *Let*

$$u_1 \ge u_2 \ge \cdots \ge u_m, \qquad v_1 \ge v_2 \ge \cdots \ge v_m,$$

where $u_i's$ and $v_i's$ are real numbers. Then

$$\sum_{i=1}^{k} u_i \le \sum_{i=1}^{k} v_i \qquad for \quad k = 1, 2, \cdots, m$$

if and only if

$$\sum_{i=1}^{m} f(u_i) \le \sum_{i=1}^{m} f(v_i)$$

for every convex increasing function f.

Proof. See [12].

□

Remark. Originally, Tomić gave a geometric proof based on a Gauss' theorem on the centroid. But it can be proved easily by using convexity and summation by parts (See [12], [15], and [16]).

Corollary 2.2. *Let* $x_1 \le x_2 \le \cdots \le x_m$ *and* $y_1 \le y_2 \le \cdots \le y_m$, *where* $x_i's$ *and* $y_i's$ *are real numbers. Then the followings are equivalent.*

(i) $\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i$ *for* $k = 1, 2, \cdots, m$

(ii) $\sum_{i=1}^{m} g(x_i) \ge \sum_{i=1}^{m} g(y_i)$ *for every convex decreasing function* g

(iii) $\sum_{i=1}^{m} h(x_i) \le \sum_{i=1}^{m} h(y_i)$ *for every concave increasing function* h.

Proof. In the above Theorem 2.1, put $u_i = -y_i$ and $v_i = -x_i$.

(i) $\Leftrightarrow$ (ii): Take $f(x) = g(-x)$ in Theorem 2.1.

(i) $\Leftrightarrow$ (iii): Take $f(x) = -h(-x)$ in Theorem 2.1. □

Now we apply the above corollary to SSD-sets.

Definition 2.3.

(i) For any positive integer n, we define

$$\mathrm{G}(\mathcal{S}, n) = \{1, 2, 2^2, \cdots, 2^{n-1}\} \quad \text{and} \quad \mathrm{G}(\mathcal{S}, \infty) = \mathrm{G}(\mathcal{S}).$$

(ii) Let A be a set and f a real valued function defined on A. Then define

$$\mu_f(A) = \sum_{a \in A} f(a)$$

if the sum of the right side exists.

Theorem 2.4. *For any SSD-set* A *we have*

(i) $\mu_g(A) \le \mu_g(G(\mathcal{S}, |A|))$ *for every convex decreasing function* g,

(ii) $\mu_h(A) \ge \mu_h(G(\mathcal{S}, |A|))$ *for every concave increasing function* h.

Proof. Let $A = \{a_1 < a_2 < a_3 < \cdots\}$. By Lemma 1.2, for all positive integers $k \le |A|$ we have

$$a_1 + a_2 + \cdots + a_k \ge 1 + 2 + 2^2 + \cdots + 2^{k-1}. \tag{2.1}$$

Now, in Corollary 2.2, put $x_i = 2^{i-1}$ and $y_i = a_i$. A direct application of the corollary with a limit argument when $|A| = \infty$ gives the result □

The following theorem, which was first proved by F. Hanson, J. M. Steele and F. Stenger (see [10]) using a variation of Ryavec's generating function method, is immediate from Theorem 2.4.

Theorem 2.5. *Let* $\{a_n\}_{n=1}^{\infty}$ *be an SSD-sequence. Then*

$$\sum_{i=1}^{m} a_i^{\,s} \le \sum_{i=1}^{m} 2^{(i-1)s} = \frac{1-2^{ms}}{1-2^{s}} \tag{2.2}$$

for all positive integers m *and all real numbers* $s \le 0$.

Proof. In Theorem 2.4 (i), put $A = \{a_1, a_2, \cdots, a_m\}$ and $g(x) = x^s$. □

At this point, one may ask whether one can have the reverse inequality of (2.2) for non-negative s. In other words, is

$$\sum_{i=1}^{m} a_i^{\,s} \ge \sum_{i=1}^{m} 2^{(i-1)s} = \frac{1-2^{ms}}{1-2^{s}} \tag{2.3}$$

for all positive integers m and all $s \ge 0$?

In view of (2.1), inequality (2.3) is true for $s = 1$. What about other positive values of s? In connection with this question we have the following theorem.

Theorem 2.6. *Let* $\{a_n\}_{n=1}^{\infty}$ *be an SSD-sequence and* β *a fixed positive real number. If*

$$\sum_{i=1}^{m} a_i^{\,\beta} \ge \sum_{i=1}^{m} 2^{(i-1)\beta} = \frac{1-2^{m\beta}}{1-2^{\beta}}$$

for all positive integers m, *then (2.3) is true for all positive integers* m, *and all* $0 \le s \le \beta$.

Proof. In Theorem 2.4 (ii), put $A = \{a_1, a_2, \cdots, a_m\}$ and $h(x) = x^{s/\beta}$. Note that $h(x)$ is concave and increasing on $x \ge 0$, whence the result follows. □

By the previous theorem, we may define λ to be the supremum of all s that satisfy (2.3) for all SSD-sequences $\{a_n\}_{n=1}^{\infty}$ and for all positive integers m. Leo Moser proved that (2.3) is true for $s = 2$ by a clever combinatorial argument (see [4, p.137] or [9, p.142, Theorem 2]). We will give an alternate analytic treatment of it in the next section. Thus we have $\lambda \geq 2$. Let

$$\{a_i\}_{i=1}^{7} = \{20, 31, 37, 40, 42, 43, 44\}. \tag{2.4}$$

By routine calculations or by using the construction from the Conway-Guy sequence (see [9], [11]), one can show the set of (2.4) has the SSD-property. Then

$$\sum_{i=1}^{7} a_i{}^s - \sum_{i=1}^{7} 2^{(i-1)s}$$

changes sign from $+$ to $-$ at $s = 3.6906742\cdots$. (Of course it is obvious from Theorem 2.6 that it cannot change sign in the opposite direction.) Hence we may conclude that $2 \leq \lambda \leq 3.6906742\cdots$.

Remark. Note that the method used by F. Hanson, J. M. Steele and F. Stenger in [10] has no immediate extension to $s \geq 0$ because of convergence problems.

We close this section with a multiplicative analogue of inequality (2.1).

Theorem 2.7. *Let $\{a_n\}_{n=1}^{\infty}$ be an SSD-sequence. Then*

$$a_1 a_2 \cdots a_m \geq 1 \cdot 2 \cdot 2^2 \cdots 2^{m-1} = 2^{m(m-1)/2}$$

for all positive integers m.

Proof. In Theorem 2.4 (ii), put $A = \{a_1, a_2, \cdots, a_m\}$ and $h(x) = \log x$. □

3. A lower bound for a_n

Let $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ be an SSD-sequence. It is an old problem to find a lower bound for a_n (see [1, pp.47-48], [2], [4], [5, p.467], [6, pp.59-60], [9] and [14]). This problem can be stated inversely:

Under the condition $a_n \le x$, find an upper bound on n. In his paper [3], N. Elkies mentioned the inter-relation between a lower bound on a_n and an upper bound on n in terms of x. A lower bound of the form

$$(3.1) \qquad a_n \ \ge \ C n^{-s} 2^n (1 + o(1))$$

corresponds to an upper bound of the form

$$(3.2) \qquad n \ \le \ \log_2 x + s \log_2 \log_2 x + \log_2 \frac{1}{C} + o(1).$$

The famous conjecture of Erdös is that (3.1) and (3.2) hold with $s = 0$ (see [8, p.64, problem C8]). In connection with the previous section, note that this conjecture of Erdös is equivalent to

$$(3.3) \qquad \sum_{i=1}^{m} {a_i}^s \ \ge \ C^s \sum_{i=1}^{m} 2^{(i-1)s} = C^s \frac{1 - 2^{ms}}{1 - 2^s}$$

for all positive integers m, for all positive s, and for some positive constant C. On the one hand $a_m \ge C 2^{m-1}$ for all m clearly implies (3.3). In the other direction, (3.3) implies $m {a_m}^s \ge C^s 2^{(m-1)s}$ or $a_m \ge m^{-(1/s)} C 2^{m-1}$; since s can be as large as desired, $a_m \ge C 2^{m-1}$ for all m. Also, Elkies showed that

$$a_n \ \ge \ 2^{-n} \binom{2n}{n} \sim \frac{1}{\sqrt{\pi}} n^{-\frac{1}{2}} 2^n$$

by an analytic method. But we point out that from L. Moser's result we can derive the better bound, $a_n > \frac{1}{\sqrt{3}} n^{-\frac{1}{2}} 2^n$ for $n \ge 2$, very easily:

$$(*) \qquad \sum_{i=1}^{n} {a_i}^2 \ \ge \ \sum_{i=1}^{n} 2^{(i-1)2} \ = \ \frac{1}{3}(4^n - 1)$$

$$\Longrightarrow \quad n {a_n}^2 \ > \ \frac{1}{3} 4^n \quad \Longrightarrow \quad a_n \ > \ \frac{1}{\sqrt{3}} n^{-\frac{1}{2}} 2^n .$$

In Theorem 3.3 below we give a new proof of Moser's result $(*)$ using an analytic method. Though Moser's proof is simple enough, we give another because it might give some further insight.

First, we need the following two lemmas. The first one is quite simple; the second is known as Laplace's method for estimating integrals.

Lemma 3.1. *Let $0 < a < b$ be fixed. Then for any $y > 0$, we have*

$$y^a + y^{-a} \le y^b + y^{-b}.$$

Proof. Note that

$$0 < y < 1 \quad \Longrightarrow \quad y^b - y^a < 0 \quad \text{and} \quad y^a y^b - 1 < 0,$$

$$y \ge 1 \quad \Longrightarrow \quad y^b - y^a \ge 0 \quad \text{and} \quad y^a y^b - 1 \ge 0.$$

Hence, for any $y > 0$, $(y^b - y^a)(y^a y^b - 1) \ge 0$. This implies that $y^a y^{2b} + y^a \ge y^{2a} y^b + y^b$ and the result follows upon dividing by y^{a+b}. □

Lemma 3.2. **(Laplace's method)** *Assume that two real valued functions $\varphi(x)$ and $f(x)$, defined on $(-\infty, \infty)$, satisfy the following four conditions:*

(i) $\varphi(x)\ (f(x))^N$ *is absolutely integrable on* $(-\infty, \infty)$, $N = 0, 1, 2, \cdots$.

(ii) $f(x) \ge 0$ *for all x and $f(x)$ attains its maximum at $x = \xi$. Furthermore*

$$\sup\{f(x) \,:\, x \in C\} < f(\xi)$$

for any closed subset C of $(-\infty, \infty)$ not containing ξ.

(iii) $f''(x)$ *exists and is continuous on $(-\infty, \infty)$ and $f''(\xi) < 0$.*

(iv) $\varphi(x)$ *is continuous at $x = \xi$, and $\varphi(\xi) \ne 0$.*

Then

$$\int_{-\infty}^{\infty} \varphi(x)\ (f(x))^N\, dx \ \sim\ \varphi(\xi)\ (f(\xi))^{N+\frac{1}{2}} \sqrt{-\frac{2\pi}{N\, f''(\xi)}}$$

as $N \longrightarrow \infty$.

Proof. See [13, Vol. I, Part II, Problem 201]. □

Theorem 3.3.

$$\sum_{i=1}^{n} a_i^{\,2} \ \geq \ \frac{1}{3}(4^n - 1)$$

Proof. Let

$$A = \left\{ \sum_{j=1}^{n} \epsilon_j a_j \ : \ \epsilon_j = +1 \ \text{ or } \ -1 \right\}$$

Clearly every element in A has the same parity and $a \in A$ implies $-a \in A$. Since $\{a_j\}_{j=1}^{\infty}$ is an SSD-sequence, we also have $0 \notin A$. Note that no integers can be expressed in more than one way in the form $\sum_{j=1}^{n} \epsilon_j a_j$, where $\epsilon_j = \pm 1$. For

$$\sum_{j=1}^{n} \epsilon_j' a_j = \sum_{j=1}^{n} \epsilon_j'' a_j, \quad \epsilon_j' = \pm 1, \quad \epsilon_j'' = \pm 1$$

is equivalent to

$$2 \sum_{\epsilon_j' = -\epsilon_j'' = 1} a_j = 2 \sum_{\epsilon_j' = -\epsilon_j'' = -1} a_j .$$

Hence $|A| = 2^n$. Now, by Lemma 3.1, we have

$$\sum_{a \in A} y^a \ \geq \ \sum_{j=1}^{2^{n-1}} \left(y^{2j-1} + y^{-(2j-1)} \right)$$

for all $y > 0$. Let $y = e^x$. Then

$$\prod_{j=1}^{n} \left(e^{a_j x} + e^{-a_j x} \right) = \sum_{a \in A} e^{a x}$$
$$\geq \sum_{j=1}^{2^{n-1}} \left((e^x)^{2j-1} + (e^x)^{-(2j-1)} \right) = \frac{\sinh(2^n x)}{\sinh x}$$

Divide by 2^n, take reciprocals, and and raise both sides to the power $2m$ to obtain

$$\left(\prod_{j=1}^{n} \frac{1}{\cosh(a_j x)} \right)^{2m} \ \leq \ 4^{mn} \left(\frac{\sinh x}{\sinh(2^n x)} \right)^{2m}$$

Then integration from $-\infty$ to ∞ yields

$$\int_{-\infty}^{\infty} \left(\prod_{j=1}^{n} \frac{1}{\cosh(a_j x)} \right)^{2m} dx \ \leq \ 4^{mn} \int_{-\infty}^{\infty} \left(\frac{\sinh x}{\sinh(2^n x)} \right)^{2m} dx . \tag{3.4}$$

To estimate the integral on the right side of (3.4), apply Lemma 3.2 with $N = 2m,\ \xi = 0$,

$$\varphi(x) = 1 \quad \text{and} \quad f(x) = \frac{\sinh x}{\sinh(2^n x)}.$$

Then, since $f(0) = 2^{-n}$ and $f''(0) = \frac{1}{3}(2^{-n} - 2^n)$,

$$\int_{-\infty}^{\infty} \left(\frac{\sinh x}{\sinh(2^n x)} \right)^{2m} dx \sim (2^{-n})^{2m+\frac{1}{2}} \sqrt{\frac{3 \cdot 2\pi}{2m(2^n - 2^{-n})}}$$

as $m \longrightarrow \infty$. Thus

$$4^{mn} \int_{-\infty}^{\infty} \left(\frac{\sinh x}{\sinh(2^n x)} \right)^{2m} dx \sim 2^{-n} \sqrt{\frac{3\pi}{m(1 - 2^{-2n})}} \tag{3.5}$$

as $m \longrightarrow \infty$. Now, in order to estimate the integral on the left side of (3.4), apply Lemma 3.2 with $N = 2m,\ \xi = 0$,

$$\varphi(x) = 1 \quad \text{and} \quad f(x) = \prod_{j=1}^{n} \frac{1}{\cosh(a_j x)}.$$

Note that here $f(0) = 1$ and $f''(0) = -\sum_{j=1}^{n} {a_j}^2$. Hence

$$\int_{-\infty}^{\infty} \left(\prod_{j=1}^{n} \frac{1}{\cosh(a_j x)} \right)^{2m} dx \sim \sqrt{\frac{\pi}{m \sum_{j=1}^{n} {a_j}^2}} \tag{3.6}$$

as $m \longrightarrow \infty$.

From (3.4), (3.5) and (3.6) it follows that

$$\sum_{i=1}^{n} {a_i}^2 \geq \frac{1}{3}(4^n - 1).$$

$\square$

Finally, we sketch how this method might be used to obtain more detailed information on the a_n. The (admittedly still rough) idea is first to obtain detailed information on all power sums $\sum_{j=1}^{n} {a_j}^{2k}$. To do this, introduce the generating function

$$\prod_{j=1}^{n} (e^{\omega_1 a_j x} + e^{\omega_2 a_j x} + \cdots + e^{\omega_{2k} a_j x})$$

where $\omega_1, \omega_2, \cdots, \omega_{2k}$ are all the $2k$-th roots of unity, and use the following generalized Laplace method:

Theorem 3.4. (Generalized Laplace Method) *Assume that a real valued function* $f(x)$ *defined on* $(-\infty,\infty)$ *satisfies the following:*

(i) $(f(x))^N$ *is integrable on* $(-\infty,\infty)$, $N=0,1,2,\cdots$.

(ii) $f(x)\geq 0$ *for all* x *and* $f(x)$ *attains its maximum at* $x=\xi$. *Furthermore*

$$\sup\{f(x)\ :\ x\in C\}\ <\ f(\xi)$$

for any closed subset C *of* $(-\infty,\infty)$ *not containing* ξ.

(iii) $f^{(2m)}(x)$ *exists and is continuous on* $(-\infty,\infty)$.

(iv) $f^{(l)}(\xi)=0$ *for* $l<2m$ *and* $f^{(2m)}(\xi)<0$.

Then

$$\int_{-\infty}^{\infty}(f(x))^N\,dx\ \sim\ (f(\xi))^{N+\frac{1}{2m}}\,\frac{\Gamma(1/(2m))}{m}\left(\frac{(2m)!}{-N\,f^{(2m)}(\xi)}\right)^{1/(2m)}$$

as $N\longrightarrow\infty$.

Proof. Imitate the proof of the Lemma 3.2 and use the fact (see [7, p.355, #3.326])

$$\int_{-\infty}^{\infty}e^{-x^{2m}}dx\ =\ \frac{\Gamma(1/(2m))}{m}.$$

□

Then take

$$f(x)\ =\ \prod_{j=1}^{n}\frac{2k}{(e^{\omega_1 a_j x}+e^{\omega_2 a_j x}+\cdots+e^{\omega_{2k} a_j x})}.$$

We have $f(0)=1$, $f^{(l)}(0)=0$ for $l<2k$ and

$$f^{(2k)}(0)\ =\ -\sum_{j=1}^{n}{a_j}^{2k}.$$

Upon applying Theorem 3.4, it follows that

$$\int_{-\infty}^{\infty}f(x)^N\,dx\ \sim\ \frac{\Gamma(1/(2k))}{k}\left(\frac{(2k)!}{N\sum_{j=1}^{n}{a_j}^{2k}}\right)^{1/2k}$$

as $N\longrightarrow\infty$.

Of course, to make this approach successful, one needs to find an appropriate upper bound for the function $f(x)$.

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