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Existence of Multiple Solutions for Second Order Ordinary Differential Equations

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Let R be the set of all real numbers. By $L^2[0,2\pi]$ we denote the space of 2π-periodic measurable functions $x:[0,2\pi]\to R$ for which $|x(t)|^2$ is integrable. The norm is given by

$$\|x\|_{L^2}=[\sum_{i=1}^{n}\int_0^{2\pi}|x_i(t)|^2dt]^{1/2}$$

By $C^k[0,2\pi]$ we denote the Banach space of 2π-periodic continuous functions $x:[0,2\pi]\to R$ whose derivatives to up order k are continuous. The norm is given by

$$\|x\|_{C^k}=\sum_{i=1}^{k}\|x^{(i)}\|_\infty$$

where $\|y\|_\infty = \sup_{t\in[0,2\pi]}|y(t)|$, the norm in $C^0[0,2\pi]$.

In this work, we prove the existence multiple solutions to the problem

$$(E_1)\qquad x''(t)+f(x(t))x'(t)+g(t,x(t))=e(t)$$

$$(B) \qquad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

where $f : R \to R,\ g : [0, 2\pi] \times R \to R$ are continuous and $e \in L^2[0, 2\pi]$, and study the existence multiple solutions to the problem

$$(E_2) \qquad x''(t) + h(t, x(t), x'(t)) + g(t, x(t)) = e(t)$$

$$(B) \qquad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

where $h : [0, 2\pi] \times R \times R \to R,\ g : [0, 2\pi] \times R \to R$ and $e : [0, 2\pi] \to R$ are continuous functions.

The existence and multiplicity of periodic solutions for some scalar forced peudulum type has been studied by several authors and we refer to [6] for classical results, and to [1] and [7] for recent results. In [4], Mawhin treats peudulum-like systems with the symmetric forcing term. In [8], Zanolin give the existence of periodic solutions of Lienard systems and the multiplicity of periodic solution for simple peudulum equations. In [2], [3], Kim, and Kim and Hirano discussed the multiple existence of Lieard system. This paper is motivated by the results in [7], and our result in this work extend some results in [7].

The proof of our results are based on coincidence degree theory and upper-lower solution method.

For the first result, we observe that if $(E_1)(B)$ has a solution, then, integrating over $[0, 2\pi]$, we obtain, if we write, for $y \in L^1[0, 2\pi]$,

$$\bar{x} = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt, \quad \tilde{x}(t) = x(t) - \bar{x}, \quad \bar{e} = \frac{1}{2\pi}\int_0^{2\pi} g(x(t))dt$$

and hence $\bar{e} \in \overline{Cog}(R)$.

From the above observation, we have the following

THEOREM 1. *Assume, besides the above conditions on f, g and e, that the following conditions hold.*
(H_1) there exists $c > 0$ such that

$$|f(x)| \geq c$$

for all $x \in R$.
(H_2) there exist real numbers $r_1, r_2, s_1, s_2, 0 < s_1 - r_1 < T, r_1 < s_2 < r_2 < s_1, A_1 \leq A_2, B_1 \geq B_2$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} g(r_1 + \tilde{x}(t))dt \leq A_1, \quad \frac{1}{2\pi}\int_0^{2\pi} g(s_1 + \tilde{x}(t))dt \geq B_1$$

$$\frac{1}{2\pi}\int_0^{2\pi} g(r_2 + \tilde{x}(t))dt \leq A_2 \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi} g(s_2 + \tilde{x}(t))dt \geq B_2$$

for every $\tilde{x}(t) \in C^1[0, 2\pi]$ having mean value zero, satisfying the boundary conditions (B) such that

$$\|\tilde{x}\|_\infty < \sqrt{\frac{\pi}{6}}\frac{\|\tilde{e}\|_{L^2}}{c}.$$

Then $(E_1)(B)$ has at least one solution if

$$A_1 \leq \bar{e} \leq B_1,$$

and $(E_1)(B)$ has at least three solutions if

$$A_2 \leq \bar{e} \leq B_2.$$

Proof. To prove our the first assertion, it is sufficient to find on open bounded set Ω in $C^1[0, 2\pi]$ such that, for each $\lambda \in (0, 1)$, the possible solutions of the problem (with $\varepsilon \in (0, 1)$ fixed)

$$(E_1^\epsilon) \qquad x'' + \lambda f(x)x' + (1 - \lambda)\varepsilon(x - \frac{r_1 + s_1}{2}) + \lambda g(x) = \lambda e(t)$$

$$(B) \qquad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

satisfy $x \notin \partial\Omega$ and such that $x'' + \varepsilon x = \frac{\varepsilon}{2}(r+s)$ has a 2π-periodic solution in Ω. To construct Ω, let $\lambda \in (0,1)$ and let x be a possible solution of $(E_1)(B)$. Multiplying the equation (E_1^ϵ) by x', integrating over $[0, 2\pi]$ and using boundary conditions (B), we have

$$\int_0^{2\pi} f(x(t))[x'(t)]^2 dt = \int_0^{2\pi} \tilde{e}(t)x'(t)dt.$$

As f is continuous and $|f|$ does not vanish, by Schwartz inequality, we have

$$\frac{\|x'\|_{L^2}\| \le \tilde{e}\|_{L^2}}{c}.$$

By Sobolev inequality,

$$\|\tilde{x}\|_\infty \le \sqrt{\frac{\pi}{6}}\frac{\|\tilde{e}\|_{L^2}}{c} \equiv M_0.$$

Let

$$\Omega^0 = \{x \in C^1[0,2\pi] | r_1 \le \bar{x} \le s_1 \ \text{ and } \ \|\tilde{x}\|_\infty \le M_0\}.$$

If x is a solution to (E_1^ϵ)(B) lying in Ω^0, then

$$x = \bar{x} + \tilde{x}.$$

Hence

$$\|x\|_\infty \le |\bar{x}| + \|\tilde{x}\|_\infty \le \max(|r_1|, |s_1|) + M_0.$$

Since f and g are continuous, we have

$$\|f\|_\infty \le M_1, \qquad \|g\|_\infty \le M_2$$

Multiplying the equation (E_1^ϵ) by x'' and integrate over $[0, 2\pi]$, we have

$$\int_0^{2\pi} [x''(t)]^2 dt$$

$$= -\lambda \int_0^{2\pi} f(x(t))x'(t)x''(t)dt + (1-\lambda)\varepsilon \int_0^{2\pi} \tilde{x}(t)x''(t)dt$$

$$-\lambda \int_0^{2\pi} g(x(t))x''(t)dt + \lambda \int_0^{2\pi} \tilde{e}(t)x''(t)dt.$$

Hence

$$\|x''\|_{L^2} \leq (\frac{M_1}{c} + 1)\|\tilde{e}\|_{L^2} + \sqrt{2\pi}(M_0 + M_2) \equiv M_3.$$

Since $\bar{x}_1 = 0$, by Sobolev inequality,

$$\|x'\|_\infty \leq \sqrt{\frac{\pi}{6}} M_3$$

for all possible solution of (E_1^ϵ)(B) lying in Ω^0. Define now Ω by

$$\Omega = \{x \in C^1[0, 2\pi] | r_1 < \bar{x} < s_1, \quad \|\tilde{x}\|_\infty < 2M_0, \quad \|x'\|_\infty \leq \sqrt{\frac{2\pi}{3}} M_3\}.$$

Suppose $x \in \partial\Omega$, then necessary $\bar{x} = r$ or s. If x satisfies (E_1^ϵ)(B), by integrating (E_1^ϵ) over $[0, 2\pi]$, we obtain

$$(1-\lambda)\varepsilon \int_0^{2\pi} (x - \frac{r_1 + s_1}{2})dt + \lambda \int_0^{2\pi} g(x(t))dt = \lambda \int_0^{2\pi} e(t)dt.$$

Hence

$$(1-\lambda)\varepsilon(\bar{x} - \frac{r_1 + s_1}{2}) + \lambda[\frac{1}{2\pi} \int_0^{2\pi} g(\bar{x} + \tilde{x}(t))dt - \bar{e}] = 0.$$

By (H_2), we have

$$(1-\lambda)\varepsilon(r_1 - \frac{r_1+s_1}{2}) + \lambda[\frac{1}{2\pi}\int_0^{2\pi} g(r_1+\tilde{x}(t))dt - \bar{e}]$$

$$\leq (1-\lambda)\varepsilon(\frac{r_1+s_1}{2}) + \lambda[A_1 - \bar{e}] < 0.$$

and

$$(1-\lambda)\varepsilon(s_1 - \frac{r_1+s_1}{2}) + \lambda[\frac{1}{2\pi}\int_0^{2\pi} g(s_1+\tilde{x}(t))dt - \bar{e}]$$

$$\geq (1-\lambda)\varepsilon(\frac{s_1-r_1}{2}) + \lambda[B_1 - \bar{e}] > 0,$$

which is impossible. Thus (E_1^ϵ),(B) has no solution on $\partial\Omega$ when $\lambda \in (0,1)$. We now are going to prove $\frac{r_1+s_1}{2} \in \Omega$. Consider the case $\lambda = 0$ in (E_1^ϵ); i.e.,

$$(E_1^{\epsilon'}) \qquad x''(t) + sx(t) = \varepsilon(\frac{r_1+s_1}{2})$$

$$(B) \qquad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0.$$

Multiply to the equation $(E_1^{\epsilon'})$ by x'' and integrate over $[0, 2\pi]$, we have

$$\int_0^{2\pi} [x''(t)]^2 dt = \varepsilon \int_0^{2\pi} [x'(t)]^2 dt$$

Since

$$\int_0^{2\pi} [x''(t)]^2 dt \geq \int_0^{2\pi} [x'(t)]^2 dt,$$

for $0 < \varepsilon < 1$ we have

$$\varepsilon \int_0^{2\pi} [x'(t)]^2 dt \geq \int_0^{2\pi} [x'(t)]^2 dt$$

Hence $x'(t) \equiv 0$ on $[0, 2\pi]$,i.e., $x \equiv c$ on $[0, 2\pi]$. Integrate the equation $(E_1^{\varepsilon\prime})$ over $[0, 2\pi]$, then

$$\varepsilon \int_0^{2\pi} x(t)dt = \varepsilon \frac{r_1 + s_1}{2} 2\pi$$

or

$$\bar{x} = \frac{\varepsilon(r_1 + s_1)}{2}.$$

Thus

$$x = \bar{x} = \frac{\varepsilon(r_1 + s_1)}{2} \in \Omega.$$

Hence $(E_1)(B)$ has at least one solution in Ω if $A_1 \leq \bar{e} \leq B_1$. We now are going to prove our second assertion. To prove the multiplicity we follow a quite similar method which is adapted in the first part.

For $\lambda \in (0, 1)$, consider the following three problems (with $\varepsilon \in (0, 1)$ fixed)

$$(Q_1) \quad x''(t) + \lambda f(x(t))x'(t) + (1 - \lambda)\varepsilon(x(t) - \frac{r_1 + s_1}{2}) + \lambda g(x(t)) = \lambda e(t)$$

$$(B) \quad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

$$(Q_2) \quad x''(t) + \lambda f(x(t))x'(t) + (1 - \lambda)\varepsilon(x(t) - \frac{r_2 + s_2}{2}) + \lambda g(x(t)) = \lambda e(t)$$

$$(B) \quad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

$$(Q_3) \quad x''(t) + \lambda f(x(t))x'(t) + (1 - \lambda)\varepsilon(x(t) - \frac{r_2 + s_1}{2}) + \lambda g(x(t)) = \lambda e(t)$$

$$(B) \qquad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0.$$

It is sufficient to find three disjoint open bounded sets Ω_1, Ω_2 and Ω_3 in $C^1[0, 2\pi]$ such that for each $\lambda \in (0,1)$, the all possible solutions to the problem (Q_1)(B), (Q_2)(B) and (Q_3)(B) satisfy respectively are not in $\partial\Omega_1, \partial\Omega_2$ and $\partial\Omega_3$. To construct Ω_1, Ω_2 and Ω_3, let $\lambda \in (0,1)$ and let x be a possible solution of (Q_1)(B). Then we have, as we did,

$$\|\tilde{x}\|_\infty \le \sqrt{\frac{\pi}{6}} \frac{\|\tilde{e}\|_{L^2}}{c} \equiv M_0.$$

Let

$$\Omega_1^0 = \{x \in C^1[0, 2\pi] | r_1 \le \bar{x} \le s_2, \quad \text{and} \quad \|\tilde{x}\|_\infty \le M_0\}.$$

If x is a solution to (Q_1)(B) lying in Ω_1^0, then

$$\|x\|_\infty \le |\bar{x}| + \|\tilde{x}\|_\infty \le \max(|r_1|, |s_2|) + M_0.$$

As we did in the first part, we have

$$\|x''\|_{L^2} \le M_3'.$$

Hence

$$\|x'\|_\infty \le \sqrt{\frac{\pi}{6}} M_3'$$

where M_3' depends only on $c, r_1, s_2, \|\tilde{e}\|_{L^2}, f$ and g for all possible solutions of (Q_1)(B) lying in Ω_1^0. Define now Ω_1 by

$$\Omega_1 = \{x \in C^1[0, 2\pi] | r_1 < \bar{x} < s_2, \ \|\tilde{x}\|_\infty < 2M_0, \ \|x'\|_\infty < \sqrt{\frac{2\pi}{3}} M_3'\}.$$

Then for all $x \in \partial\Omega_1$, as we did in the first part, x does not satisfy the equation and, for $\lambda = 0$, we have $x = \bar{x} = \frac{r_1+s_2}{2} \in \Omega_1$. Therefore the problem $(E_1)(B)$ has at least one solution in Ω_1.

Similarly, we can construct open bounded set

$$\Omega_2 = \{x \in C^1[0,2\pi] | s_2 < \bar{x} < r_2,\ \|\tilde{x}\|_\infty < 2M_0, \|\tilde{x}\|_\infty < \sqrt{\frac{2\pi}{3}} M_3''\}$$

where M_3'' depends only on $c, r_2, s_2, \|\tilde{e}\|_{L^2}, f$ and g, and

$$\Omega_3 = \{x \in C^1[0,2\pi] | r_2 < \bar{x} < s_1,\ \|\tilde{x}\|_\infty < 2M_0,\ \|\tilde{x}\|_\infty < \sqrt{\frac{2\pi}{3}} M_3'''\}$$

where M_3''' depends only on $c, r_2, s_1, \|\tilde{e}\|_{L^2}, f$ and g, in which the problem $(E_1)(B)$ has at least one solution in Ω_2 and Ω_3. Since $\Omega_i \cap \Omega_j = \emptyset$, $i, j = 1, 2, 3$, $i \neq j$, we has three distinct solutions.

Example 1. Suppose f satisfies the conditions of theorem, $e \in L^2[0, 2\pi]$, and $a, b > 0$. Then if

$$c \geq \sqrt{\frac{\pi}{6}} \frac{\|\tilde{e}\|_{L^2}}{\min_{i=1,2} d_i}$$

where

$$d_1 = \min\{\frac{|s_2 - r_1|}{2}, \frac{|r_1 - s_1 + 2\pi|}{2}\}, \qquad d_2 = \min\{\frac{|r_2 - s_2|}{2}, \frac{|s_1 - r_2|}{2}\},$$

and r_1 and s_1 are respectively points at which $a\,sin\,x + b\,sin\,2x$ has maximum and minimum such that

$$[a\,sin\,r_1 + b\,sin\,2r_1] = -[a\,sin\,s_1 + b\,sin\,2s_1],$$

and r_2, s_2 are respectively points at which $a\,sin\,x + b\,sin\,2x$ has relative maximum and relative minimum such that

$$[a\,sin\,r_2 + b\,sin\,2r_2] = -[a\,sin\,s_2 + b\,sin\,2s_2].$$

Then the problem

$$x'' + f(x)x' + [a\,sin\,x + b\,sin\,2x] = e(t)$$

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

has at least one solution x with $\bar{x} \in [s_1 - 2\pi, r_1]$, one solution x with $\bar{x} \in [r_1, r_2]$, one solution x with $\bar{x} \in [s_2, r_2]$ and one solution x with $\bar{x} \in [r_2, s_1]$.

To prove our second arrertion, we assume

$$h(t, x, 0) = 0$$

for every $(t, x) \in [0, 2\pi] \times R$ and that there exists some $T > 0$ such that

$$g(t, x + T) = g(t, x)$$

for every $(t, x) \in [0, 2\pi] \times R$.

We will say that h in problem $(E_2)(B)$ satisfies Nagumo-type condition on $[r, s]$ if there exists a constant $c > 0$ such that for each $\lambda \in [0, 1]$ and each possible solution of

(E_2^λ) $$x''(t) + \lambda h(t, x(t), x'(t)) + \lambda g(t, x(t)) = \lambda e(t)$$

(B) $$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

satisfying $r \le x(t) \le s,\ t \in [0, 2\pi]$, we have

$$\|x'\|_\infty < c.$$

Examples of admissible h are the following ones:

a) h depends only on x'

b) $|h(t,x,y)| \leq \gamma(|y|)$ for $(t,x,y) \in [0,2\pi] \times [r,s] \times R$ where γ is positive, continuous and such that

$$\int_0^\infty \frac{sds}{\gamma(s)} = +\infty.$$

Now we have the following

THEOREM 2. *Assume, besides the above conditions on h and g there exists there exists real numbers r_1, r_2, s_1, s_2 with $r_1 < s_2 < r_2 < s_1$ and $0 < s_1 - r_1 < T$ such that*

$$g(s_1) \leq g(s_2), \qquad g(r_2) \leq g(r_1)$$

and h satisfies Nagumo type condition on $[s_1 - T, s]$. Then $(E_2)(B)$ has at least one solution if for all $t \in [0,2\pi]$,

$$(\mathrm{I}_1) \qquad g(t,s_1) \leq e(t) \leq g(t,r_1),$$

and $(E_2)(B)$ has at least two solutions not differing by a multiple of T if, for all $t \in [0,2\pi]$,

$$(\mathrm{I}_2) \qquad g(t,s_1) < g(t,s_2) \leq e(t) \leq g(t,r_2) < g(t,r_1),$$

and $(E_2)(B)$ has at least four solutions not differing by a multiple of T if strict inequalities holds in (I_2).

Proof. Suppose (I_1). Then, for all $t \in [0,2\pi]$, we have

$$e(t) - g(t, r_1 + kT) - h(t, r_1 + kT, 0) = e(t) - g(t, r_1) \leq 0$$

$$e(t) - g(t, s_1 + JT) - h(t, s + JT, 0) = e(t) - g(t, s_1) \geq 0$$

with strict inequalities if they hold in (I_1). Hence by Mawhin's classical results, there exists, by taking $k = J = 0$, at least one solution $x_1(t)$ of $(E_2)(B)$ such that $r_1 \le x(t) \le s_1$.

Now suppose the strict inequalities holds; i.e., for all $t \in [0, 2\pi]$.

$$(\mathrm{I}'_1) \qquad g(t, s_1) < e(t) < g(t, r_1).$$

Now, if we define

$$L : D(L) \subseteq C^1[0, 2\pi] \longrightarrow C^0[0, 2\pi]$$
$$x \longrightarrow x''$$

where $D(L) = C^2[0, 2\pi]$ and

$$N : C^1[0,2\pi] \longrightarrow C^0[0, 2\pi]$$
$$x \longrightarrow x''.$$

Then L is a Fredholm mapping of index zero and N is L-completely continuous.

Let

$$\Omega_{k,J} = \{x \in C^1[0, 2\pi] |\, r_1 + kT < x(t) < s_1 + jT \text{ for } t \in [0, 2\pi] \text{ and } \|x'\|_\infty < c\}.$$

Then the boundary value problem $(E_2^\lambda)(B)$ becomes

$$Lx - \lambda Nx = 0, \qquad \lambda \in [0, 1]$$

and when the strict inequalities hold in (I_1), the following coincidence degree exist and have the corresponding values, where exist and have the corresponding values, where d_B denotes the Brouwer degree, and

$$D_L(L - N, \Omega_{0,0}) = d_B(\Gamma, (r_1, s_1), 0) = +1$$

$$D_L(L-N,\Omega_{-1,-1}) = d_B(\Gamma,(r_1-T,s_1-T),0) = +1$$

$$D_L(L-N,\Omega_{-1,0}) = d_B(\Gamma,(r_1-T,s_1),0) = +1$$

where $(Tu)(t) = \frac{1}{2\pi}\int_0^{2\pi}[e(t)-g(t,u(t))]dt$. But

$$\Omega_{0,0}\cap\Omega_{-1,-1} = \emptyset$$

and

$$\Omega_{0,0}\subseteq\Omega_{-1,0}, \qquad \Omega_{-1,-1}\subseteq\Omega_{-1,0}.$$

So that the excision property of degree implies

$$\begin{aligned} 1 = D_L(L-N,\Omega_{-1,0}) =& D_L(L-N,\Omega_{-1,-1},0) \\ &+ D_L(L-N,\Omega_{0,0},0) \\ &+ D_L(L-N,\Omega_{-1,0}\backslash(\bar{\Omega}_{-1,-1}\cup\bar{\Omega}_{0,0})) \\ =& 2 + D_L(L-N,\Omega_{-1,0}\backslash(\bar{\Omega}_{-1,-1}\cup\bar{\Omega}_{0,0})). \end{aligned}$$

Hence

$$D_L(L-N,\Omega_{-1,0}\backslash(\bar{\Omega}_{-1,-1}\cup\bar{\Omega}_{0,0})) = -1.$$

Hence there exists a solution x_2 such that, for all $t\in[0,2\pi]$, $r_1-T<x_2(t)<s_1$, $x_2(\tau)>s_1-T$ for some $\tau\in[0,2\pi]$ and $x_2(\tau')<r_1$ for some $\tau'\in[0,2\pi]$.

Consequently, this solution cannot differ from the one in $\Omega_{0,0}$ by a multiple of T. Hence $(E_2)(B)$ has at least two solutions not differing by a multiple of T if (I'_1) holds.

Now suppose (I_2). Then, for all $t\in[0,2\pi]$, we have

$$e(t)-g(t,s_1)-h(t,s_1,0) = e(t)-g(t,s_1) > 0,$$

$$e(t)-g(t,r_2)-h(t,r_2,0) = e(t)-g(t,r_2) \le 0.$$

Hence, there exists at least one solution $x_1(t)$ of $(E_2)(B)$ such that $r_2 \leq x_1(t) \leq s_1$ for all $t \in [0, 2\pi]$. Again, for all $t \in [0, 2\pi]$, we have

$$e(t) - g(t, s_2) - h(t, s_2, 0) = e(t) - g(t, s_2) \geq 0,$$

$$e(t) - g(t, r_1) - h(t, r_1, 0) = e(t) - g(t, r_1) < 0.$$

Therefore, there exists at least one solution $x_2(t)$ of $(E_2)(B)$ such that $r_1 \leq x_2(t) \leq s_2$ for all $t \in [0, 2\pi]$. Since $r_1 < s_2, r_2, s_1$, two solutions are different and moreover two solutions can not differ from by a multiple of T because $0 < s_1 - r_1 < T$. Since $g(t, s_1) < e(t) < g(t, r_1)$,as we did by the coincidence degree, we have a solution x_3 such that, for all $t \in [0, 2\pi]$, $r_1 - T < x_3(t) < s_1$, $x_3(\tau) > s_1 - T$ for some $\tau \in [0, 2\pi]$ and hence $x_3(\tau) > s_2 - T$, and $x_3(\tau') < r_1$ for some $\tau' \in [0, 2\pi]$ and hence $x_3(\tau') < r_2$. Therefore the third solution can not differ from x_1, x_2 in $\Omega_{0,0}$ by a multiple of T.

Consequently, there exist at least three solutions of $(E_2)(B)$ not differing by a multiple of T.

Now suppose the strict inequalities hold;i.e., for all $t \in [0, 2\pi]$,

$$(\mathrm{I'}_2) \qquad g(t, s_1) < g(t, s_2) < e(t) < g(t, r_2) < g(t, r_1).$$

Note that, for all $t \in [0, 2\pi]$, we have

$$e(t) - g(t, s_i + kT) - h(t, s_i + kT, 0) = e(t) - g(t, s_i) > 0$$

$$e(t) - g(t, r_i + jT) - h(t, r_i + jT, 0) = e(t) - g(t, r_i) < 0, \quad i = 1, 2.$$

Then clearly $(E_2)(B)$ has three solutions $x_1(t), x_2(t)$ and $x_3(t)$ such that $r_1 \leq x_1(t) \leq s_2$, $s_2 \leq x_2(t) \leq r_2$ and $r_2 \leq x_3(t) \leq s_1$, for all $t \in [0, 2\pi]$, and they are distinct and each of them are not differing by a multiple of T. For our fourth solution. Let

$$\Omega_{k,J}^{<i,j>} = \{x \in C^1[0, 2\pi] |\, r_i + kT < x(t) < s_j + jT,\, t \in [0, 2\pi],\, \|x'\|_\infty < c\}$$

$$\Omega^{[i,j]}_{k,J} = \{x \in C^1[0,2\pi] | s_i + kT < x(t) < r_j + jT,\ t \in [0,2\pi],\ \|x'\|_\infty < c\}$$

$(k \leq 1)$, where c is constant given by Nagumo condition. But $\Omega_1 = \Omega^{<1,2>}_{-1,-1}$, $\Omega_2 = \Omega^{[2,2]}_{-1,-1}$, $\Omega_3 = \Omega^{<2,1>}_{-1,-1}$, $\Omega_4 = \Omega^{<1,2>}_{0,0}$, $\Omega_5 = \Omega^{[2,2]}_{0,0}$, $\Omega_6 =^{<2,1>}_{0,0}$ are mutually disjoint subset of $\Omega^{<1,1>}_{-1,0}$ and

$$D_L(L - N, \Omega^{<1,1>}_{-1,0}) = d_B(\Gamma, (r_1 - T, s_1), 0) = +1$$

$$D_L(L - N, \Omega_1) = d_B(\Gamma, (r_1 - T, s_2 - T), 0) = +1$$

$$D_L(L - N, \Omega_2) = d_B(\Gamma, (s_2 - T, r_2 - T), 0) = -1$$

$$D_L(L - N, \Omega_3) = d_B(\Gamma, (r_2 - T, s_1 - T), 0) = +1$$

$$D_L(L - N, \Omega_4) = d_B(\Gamma, (r_1, s_2), 0) = +1$$

$$D_L(L - N, \Omega_5) = d_B(\Gamma, (s_2, r_2), 0) = -1$$

$$D_L(L - N, \Omega_6) = d_B(\Gamma, (r_2, s_1), 0) = +1.$$

Hence, by the excision property of degree,

$$1 = D_L(L - N, \Omega^{<1,1>}_{-1,0}) = 2 + D_L(L - N, \Omega^{<1,1>}_{-1,0} \setminus \cup_{1 \leq i \leq 6} \bar{\Omega}_i).$$

Therefore

$$D_L(L - N, \Omega^{<1,1>}_{-1,0} - \cup_{1 \leq i \leq 6} \bar{\Omega}_i) = -1.$$

Consequently, $(E_2)(B)$ has a solution x_4 in $\Omega^{<1,1>}_{-1,0} - \cup_{1 \leq i \leq 6} \bar{\Omega}_i$, i.e., a solution such that $r_1 - T < x(t) < s_1$ for all $t \in [0, 2\pi]$, $x_4(\tau_1) > s_2 - T$, $x_4(\tau_2) < s_2 - T$, $x_4(\tau_3) > r_2 - T$, $x_4(\tau_4) < r_2 - T$, $x_4(\tau_5) > s_1 - T$, $x_4(\tau_6) < r_1$, $x_4(\tau_7) > s_2$, $x_4(\tau_8) < s_2$, $x_4(\tau_9) < r_2$, $x_4(\tau_{10}) > r_2$ for some $\tau_1, \tau_2, \cdots, \tau_{10} \in [0, 2\pi]$. Thus this solution x_4 can not differ from x_1, x_2, x_3 by a multiple of T.

Example 2. Suppose h is a function satisfying the assumption above and Nagumo condition on $[r_1 - 2\pi, 2\pi - r_1]$ where r_1 is the point at which $a\,sin\,x + b\,sin\,2x$ has its maximum value. Let $r_2 \in [0, 2\pi]$ be a point at which $a\,sin\,x + b\,sin\,2x$ has it relative maximum such that $g(r_2) < g(r_1)$. Then the boundary value problem

$$x''(t) + h(t, x(t), x'(t)) + [a\,sin\,x + b\,sin\,2x] = e(t)$$

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

has at least one solution if $\|e\|_\infty \leq a\,sin\,r_1 + b\,sin\,2r_1$, at least two solutions not differing by a multiple of 2π if $\|e\|_\infty < a\,sin\,r_1 + b\,sin\,2r_1$, at least three solutions not differing by a multiple of 2π if $\|e\|_\infty \leq a\,sin\,r_2 + b\,sin\,2r_2$ and at least four solutions not differing by a multiple of 2π if $\|e\|_\infty < a\,sin\,r_2 + b\,sin\,2r_2$.

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